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TEACHING STATISTICAL INFERENCE IN ELEMENTARY MATHEMATICS COURSES*

S. S. WILKS, Princeton University

I am delighted with this opportunity to contribute to the ferment now going on concerning the underclass mathematics curriculum. At first I thought there was probably no surer way to increase the turmoil than to throw in a few thoughts and suggestions about including some statistical inference topics in redesigned freshman and sophomore mathematics courses! However, after carefully examining some of the new material which has been prepared at the freshman level, particularly *Universal Mathematics* by the Duren Committee of the Mathematical Association, *Principles of Mathematics* by Allendoerfer and Oakley, and *Finite Mathematics* by Kemeny, Snell and Thompson, I think the stage is already set for inserting some of the basic ideas of statistical inference. In fact, with the excellent introductions to the theory of sets and probability in these books the job is already under way! Statistical inference is so thoroughly intertwined with probability theory that I expect to be using the phrase "probability and statistics" about as frequently as "statistical inference."

In discussing this topic I shall divide my remarks into five parts. First, a short discussion of the substance and origin of statistical inference. Second, how the ideas are now being passed on in our educational system. Third, when and where the elementary concepts of statistical inference should be taught. Fourth, a suggested set of statistical inference ideas for the main freshman-sophomore sequence of mathematics courses. Fifth, summary and conclusions.

1. What is statistical inference and where did it come from? Statistical inference is a body of concepts and mathematical methods for making inferences about populations of objects from random samples drawn from these populations. These populations occur in all fields of inquiry. We have all sorts of populations of human beings and other living organisms, mass-produced articles, transactions, and repeated experiments, trials and tests. The inferences come in the form of statements with specified margins of uncertainty expressed in terms of probability. The statements refer to numerical characteristics of the populations derived from information contained in samples drawn from the populations. This field has been developed during the last thirty or forty years in the course of dealing with problems of experimentation and investigation in many different fields of science, industry, business and government. The body of specific results and methods which has thus grown up is large and widely scattered in the literature of the natural and social sciences and technology. The underlying ideas and concepts, however, are relatively few in number and they are essentially mathematical. The key results of statistical inference to date, in most cases, have been obtained by persons highly com-

* Invited address at the Thirty-eighth Summer Meeting of the Mathematical Association of America, August 28, 1957.

petent in mathematics working closely with scientists and investigators in these various fields. These results are published largely in the journals of statistics and mathematical statistics, of which there are six or seven important ones in the world. Applications of basic results in statistical inference and adaptations of them to specific problems in a given field are usually made by investigators in that field and are incorporated in their research or technical papers.

2. How has statistical inference been handled by our educational system? At the high school level, education in topics in this area is virtually nonexistent, although the new College Entrance Examination Board Commission on Mathematics is now taking the initiative in getting some of it started. At the college and university level knowledge in this area has been transmitted mainly outside of mathematics departments through *introductory* statistics courses at the upper class and graduate levels in economics, sociology, psychology, biology, and other departments or schools. The method of teaching even the most basic concepts at both levels in any of these departments is largely intuitive and nonmathematical. The ideas are almost completely couched in the language of illustrative examples peculiar to that particular department. The result is that the student does not learn these fundamental concepts in a precise manner and with an appreciation of their full generality.

Let me say, however, that these statistics courses are steadily improving because of the better statistical training being received by the younger men in those departments who are now teaching these courses. But I do not believe that the problem of teaching the basic ideas of statistical inference will ever be satisfactorily solved by introductory statistics courses taught in this intuitive manner at the upper class or graduate level in departments of a college or university scattered all over the campus.

In the first place the persons teaching these courses have, in general, a secondary interest in statistics. Their primary competence and interest is usually in some substantive area within their department. Their scholarship and research interests, as far as statistical inference is concerned, tend to be in the application of ready-made statistical methods to problems in their own field.

In the second place there is a fundamental need to teach the basic ideas of probability and statistical inference early in college—some of them even in high school. Teachers in the various departments and schools who are competent and interested in statistics would then be in a much better position to introduce students majoring in these departments to the important nonmathematical as well as mathematical aspects of the real statistical problems and methods peculiar to those departments. There are plenty of statistical problems and methods in economics, psychology, sociology, and biology to justify the existence of non-introductory statistics courses of special interest to students in those fields. As a matter of fact, the same can be said of departments of physics, chemistry and engineering, whose students have had even less training in probability and sta-

tistics because of their dependence on the traditional freshman and sophomore mathematics courses for the basic mathematical training of these students. But I will say more on this subject later.

I do not think it is an overstatement to say that probability and statistical concepts are involved in the problems and thinking of modern scientific and technological society as much as or more than any other body of mathematical ideas. At one end of the spectrum we find the average citizen confronted with the intelligence scores of his children, insurance problems, advertising and sales claims, public opinion polls, *etc.* He should be introduced to at least the rudiments of probability and statistics at the high school level. At the other end of the spectrum there is the scientist in almost every field who designs his experiments and analyzes and interprets the results by probability and statistical methods. Yet the situation we now have in the teaching of the fundamentals of probability and statistics could hardly be more chaotic. There is virtually none of it at the high school level. At the college level the situation, as I have tried to describe, can perhaps best be summarized by saying that it is similar to what we would have if every department in the university which needed chemistry for its curriculum taught its own introductory chemistry course. Or if every department which needed mathematics taught its own introductory mathematics course. This is justifiable for the later and more specialized courses heavily loaded with the subject matter of that department, but it would be ridiculous to handle the general introductory courses in this manner.

3. When and where should the elementary concepts of statistical inference be taught? In my opinion the basic concepts of statistical inference should be taught in the freshman and sophomore years in college. Some of the more elementary ideas of probability and statistics should eventually be taught in high school mathematics. I have some thoughts on this subject but I will not discuss them here.

By getting these ideas early in college the student in the natural and social sciences and engineering will be in a position to benefit from them in some of his upper class courses, reading or experimental work, in much the same way that students of physics and engineering now utilize their knowledge of traditional freshman and sophomore mathematics in the work of their upper class years.

Since the ideas of probability and statistics occur in so many different fields they should be taught to a wide range of students—not only to students interested in going into the social sciences but into the natural sciences and engineering, as well as those planning to go into business schools, medical schools and even law.

One of the outstanding examples of an unfulfilled need for some training in probability and statistics is in engineering education. The statistical quality control movement which started during the war and which has now pretty well permeated American industry caught the engineering profession completely untrained in even the rudiments of probability and statistics. In most places the

engineering schools rely on mathematics departments for their foundational mathematical training and the traditional courses contained no probability and statistics. The American Society for Quality Control, which was organized to promote this movement, has devoted a tremendous amount of effort, in cooperation with individuals in many colleges and universities of the country, trying to meet the minimum training needs in probability and statistics for practicing engineers. While the educational program in engineering schools is being improved to meet this need it is far from satisfactory. The main trouble is that special probability and statistics courses for engineers are upper class electives in most colleges and universities and are taken by a relatively small fraction of engineering students. Many of the engineering students who do not take such a course find it necessary to learn something about the subject somehow or other immediately after they get into industry. Whenever a course in engineering probability and statistics is given it is found necessary to spend at least a fourth of a semester on basic material on probability and statistical inference before really getting into the topics of particular engineering interest. This basic material should have been taken at the freshman-sophomore level.

If the basic concepts of probability and statistics are to be taught to a wide range of students, then where should they be taught? In my opinion they should be taught in the major freshman-sophomore sequence of mathematics courses. That is the sequence now required of students going into engineering and the physical sciences. Revision of these courses along the lines of the three books I have already mentioned together with the inclusion of some topics in statistical inference would make them better foundational mathematics courses for students who intend to go into the physical sciences and engineering. Not only that, it would also make them good basic courses for students intending to go into the social and biological sciences and it could be expected that eventually departments in these sciences would recommend or require their students to take the revised courses.

Some persons will raise the question at this point as to whether the basic concepts of probability and statistics should not be taught at the freshman level in a statistics department. The creation of statistics departments is a fairly recent development in some of the larger universities. The success with which this movement extends to the smaller colleges and universities will depend inversely on how well the mathematics departments meet the growing needs for the teaching of statistics in those institutions. Almost all the work being done by the relatively few statistics departments now in existence is at the upperclass and graduate levels and, in general, the work is excellent. However, it will be some time before they are in a position to operate effectively at the underclass level. Even when they develop underclass programs they will have to depend on mathematics departments to provide foundational training in such topics as set theory, analytic geometry and calculus or do it themselves, which would be an unjustifiable duplication of effort. This means, in effect, that it may be awkward for a statistics department to begin its teaching program before the

second term freshman or first term sophomore year. But until the statistics departments now in existence, or which will be established, do set up underclass programs this would not vitiate the argument that some of the basic concepts of statistical inference should be taught in the major freshman-sophomore sequence in the mathematics departments of the hundreds of smaller colleges of the country which do not have separate statistics departments. Such a program would provide the best potential now, and in the foreseeable future, for the country as a whole, for disseminating these ideas to the broadest possible range of college students who can benefit from them. Some may argue that this material is unfamiliar to mathematics teachers and hence cannot be effectively taught by them. It may be unfamiliar, but our experience at Princeton is that it can be quickly picked up. I am confident that good mathematics teachers can master this material during the course of a summer institute such as those which have been sponsored by the National Science Foundation during the last two summers. Needless to say, the teacher who can best teach these probability and statistics concepts at the freshman and sophomore level is like the teacher who can best teach the concepts of differential and integral calculus at this level: he not only has a firm grasp of the mathematical concepts and results but he has a genuine feeling and appreciation for good examples and illustrations. There is a great abundance of such examples suitable for illustrative purposes in probability and statistics.

4. What topics in statistical inference should be taught in elementary college mathematics courses? Any treatment of statistical inference concepts at the freshman level must be preceded by at least two or three weeks of elementary probability theory which must in turn be preceded by a similar period of treatment of elementary set theory. Excellent presentations of these two topics are given in the three books mentioned at the beginning of this paper. In such a treatment the student should come out with a good idea of a finite sample space (the set of all possible outcomes of an operation or experiment) and how to set up such spaces in a variety of problems. He should know something about events and their manipulation and representation by Venn diagrams; probability measure, conditional probability, independence, rules for determining probabilities of unions and intersections of events from the probabilities of the constituent events; evaluation of probabilities of events in problems whose sample points are assigned equal probabilities; random numbers and their use as a mechanism for experimentally realizing specified probabilities on a finite sample space.

The coverage of elementary probability should be extended to a thorough discussion of the concept of a random variable or chance variable and its associated distribution function, with a variety of illustrative examples. Or we can think of this extension as an introduction to statistical inference. Particular attention should be paid at least to some simple cases of the hypergeometric distribution which occurs when drawing from finite populations without replace-

ment; the binomial distribution which occurs in repeated independent trials and how this distribution is a limiting case of the hypergeometric distribution; and the Poisson distribution as a certain limiting case of the binomial distribution. These distributions and their applications should be well illustrated with examples from games of chance, genetics, psychology, bacterial and radioactive counting, traffic, industrial sampling and other fields. Abbreviated tables, particularly of the binomial and Poisson distributions, could be inserted at this point to facilitate the working of exercises. These distributions are all associated with discrete random variables which naturally arise from finite sample spaces. The concept of continuous chance variables and their distributions, at least in cumulative form, should be introduced and illustrated with simple geometric examples. While a bit of calculus would be helpful at this point it is really not yet necessary if the cumulative form of the distribution is used. The normal distribution, together with brief tables, should be introduced at this point as an approximation to the binomial distribution even though the approximation cannot be rigorously developed at this level.

The next ideas we need to introduce are those of the mean and variance of the distribution of a random variable. For the case of discrete random variables these are straightforward and the definitions can be applied to find the mean and variance of random variables having the hypergeometric, binomial and Poisson distributions by a little algebraic manipulation. However, a full understanding of the mean and variance of a continuous random variable and their application to even the simpler distributions requires elementary integral calculus and should be inserted at the appropriate point in the freshman-sophomore sequence.

Having defined the mean and variance of a discrete random variable it is a short step by an elementary argument to establish the Chebyshev inequality which states that for any positive λ the probability is at least $1-1/\lambda^2$ that a chance variable differs from its mean by less than $\lambda\sigma$, where σ is the square root of the variance of the random variable. Application of the Chebyshev inequality to the binomial distribution yields the result that if p is the probability of a success on a single trial and if x is the number of successes in n independent trials then for any positive ϵ the probability is at least

$$1 - \frac{p(1-p)}{\epsilon^2 n}$$

that the relative frequency x/n differs from p by less than ϵ. The student is then introduced to a simple case of the law of large numbers, namely, that by taking n sufficiently large the probability that x/n lies within the interval $(p-\epsilon, p+\epsilon)$ can be made arbitrarily near 1.

Furthermore, since p lies on the interval (0, 1), $p(1-p)$ has a maximum of 1/4. Hence, the probability is at least $1-1/(4\epsilon^2 n)$ that x/n differs from p by less than ϵ. This is equivalent to stating that for any $\epsilon>0$ the probability is at least $1-1/(4\epsilon^2 n)$ that p, even if it is unknown, is bracketed by the observable

interval $(x/n-\epsilon, x/n+\epsilon)$. Thus, we have the concept of a confidence interval for an unknown population parameter introduced in a natural setting at a very elementary level. If we wish to approximate the probability that this interval brackets p, calculations must be made from tables of the binomial distribution. For large n the student can be told how to approximate the probability by using the normal distribution with a comforting remark that the proof of this procedure depends on methods of analysis beyond the scope of the course!

Equipped with the concepts already referred to, the student is now in a position to consider the problem of sampling from finite populations. More precisely, suppose every object in a population of N objects has a value of a variable x. Consider all possible samples of n objects which could be drawn from the population. Every sample has a mean $\bar{x}$; and all possible sample means have a distribution. If the drawing of a sample is such that all possible samples are forced to be equally probable it follows by elementary algebraic methods that the mean of this distribution is exactly the same as the mean μ of the x's of all objects in the population and that the variance of the distribution is $\sigma^2(1/n-1/N)$, where σ^2 is the variance of the x's of all objects in the population, all equally weighted. For very large N relative to n the student will see that the variance of the sample means is approximately σ^2/n.

The problem of the mean and variance of the distribution of means of samples of size n for indefinitely large N can also be approached by setting up product sample spaces and probability distributions associated with independent random variables. Thus, in general it is an elementary operation to show that if x_1 and x_2 are two independent discrete random variables having means μ_1 and μ_2 and variances σ_1^2 and σ_2^2, then $a_1x_1+a_2x_2$ is a random variable whose distribution has mean $a_1\mu_1+a_2\mu_2$ and variance $a_1^2\sigma_1^2+a_2^2\sigma_2^2$. The result can be extended to n independent discrete random variables by induction. (The analogous results for the case where x_1 and x_2 are independent continuous random variables depends, of course, on integral calculus.) In particular, if we have independent discrete random variables $(x_1, \cdots, x_n)$ all having means equal to μ and variances equal to σ^2, then $(x_1, \cdots, x_n)$ is called a sample of size n from an infinite population having mean μ and variance σ^2. By elementary methods the student can be shown that the mean value of the sample mean $\bar{x}(=(x_1+\cdots+x_n)/n)$ is μ, the population mean, and the variance of $\bar{x}$ is σ^2/n. He can be told at this stage that if the population is normal, the distribution of $\bar{x}$ is normal with mean μ and variance σ^2/n, with a promise of proof in sophomore calculus. He can also be told that if n is large $\bar{x}$ has an approximately normal distribution, but that the proof, which he could expect to encounter as an upperclassman or graduate student, requires nonelementary methods of analysis.

There are, of course, other applications of the results concerning the mean and variance of a linear function of two or more independent random variables. Some of these can be handled as examples and exercises. An important case is the problem of the mean and variance of the difference of means of samples from two different populations. Another important case is the sample sum

$x_1+x_2+\cdots+x_n$ whose distribution the student will readily find to have mean $n\mu$ and variance $n\sigma^2$. A variety of simple and interesting exercises can be devised concerning sample means and sample sums.

After a discussion of the distribution theory of sample means and sample sums it is a natural and easy step to spend a class hour or two on confidence intervals for population means. As in the case of the binomial distribution the Chebyshev inequality provides an elementary introduction to the subject. Thus, if the population variance σ^2 is known it is found that for any given $\epsilon>0$ one can immediately state that the probability has a value at least equal to

$$1-\frac{\sigma^2}{\epsilon^2}\left(\frac{1}{n}-\frac{1}{N}\right)$$

that the population mean μ is bracketed by the interval $(\bar{x}-\epsilon, \bar{x}+\epsilon)$. From this an elementary case of the law of large numbers will be evident. The student can be told how to approximate the probability that μ is bracketed by this interval by using the normal distribution, at the same time assuring him that the method can be rigorously established by methods of analysis beyond the course. I am sure it will be some time before we can honestly pretend to present a rigorous proof of the central limit theorem, even in the case of simple random sampling, to freshmen and sophomores! If σ is unknown he can also be told in a similar vein, without proof, about the possibility of replacing σ^2 by the sample variance s^2 in the procedure for determining approximate confidence limits and obtain results still valid for large n. At this juncture he should be told something about the "Student" t ratio which is so widely used in problems of applied statistics and how confidence limits for μ are obtained from it. The underlying assumptions can be precisely stated but the proof is definitely nonelementary. Finally, before leaving the subject of sample means some attention should be paid to the problems of obtaining confidence limits of the difference of two population means which is so important in simple experiments involving an experimental and a control group which occurs in many fields of science.

The next basic concept in statistical inference which can now be introduced is the idea of testing a statistical hypothesis. An experimental setup involving repeated independent trials, each ending only in success or failure, is an ideal starter for discussing this topic. The distribution which yields all the necessary probabilities is, of course, the binomial distribution. In this case one can easily set up the mathematical structure for interpreting the results of simple experiments for such problems as whether a medical or agricultural treatment has an effect, whether a die is biased or whether a person can discriminate between two brands of beer or cigarettes. Examples of testing whether lots of mass-produced articles have more than an allowable fraction of defectives can be devised for hypothesis-testing involving the Poisson distribution. As far as problems involving sample means are concerned, there is a very short bridge between confidence interval theory and hypothesis-testing. The classical Pearson chi-square test

could be introduced and discussed briefly at this point but without proof.

In discussing statistical tests the notion of the power of the test is basic and should be brought out. The ideas of the power of a statistical test can be historically and most effectively introduced by considering an industrial acceptance sampling plan whereby a lot of size N having Np defectives is accepted if a random sample of size n has c or fewer defectives. The exact probability of acceptance for any N, n and c can be determined from a hypergeometric distribution and illustrated for small N, n and c. For large values of N and n, and small p the student can be shown that this probability can be approximated satisfactorily by the Poisson distribution. The probability of acceptance as a function of p, the fraction of defectives in the lot, called the operating characteristic function of the sampling plan, provides an excellent introduction to the concepts of producer's risk (probability of rejecting lots of satisfactory quality) and consumer's risk (probability of accepting lots of unsatisfactory quality). Extension of the ideas of producer's and consumer's risks to the fundamental type I and type II errors in any problem of hypothesis-testing is now straightforward and can be illustrated by a variety of examples based on problems involving the binomial, Poisson, and normal distributions. The operating characteristic function and the extension of this idea to any statistical test becomes a natural vehicle for introducing the basic idea of statistical decision theory which might be discussed briefly.

Finally, it is my opinion that some of the more important rudimentary statistical inference concepts involved in problems of two or more random variables should be introduced in underclass mathematics courses. The idea of a regression function, that is the mean value of one random variable for a specific value of the other random variables, is basic. Particular attention should be paid to the case of two random variables and linear regression functions. Estimation of regression coefficients and regression functions by the method of least squares, especially for the linear case, is important. This of course involves differential calculus. Some simple problems of hypothesis testing in problems of pairs of random variables should be included. The problem of testing the independence of two random variables from the information in a sample should be mentioned. The rank correlation test is simple enough to be rigorously developed at this level. The two by two contingency table should also be discussed briefly.

Almost all of the statistical inference topics I have indicated together with the necessary probability background have been taught in a special second-term freshman course at Princeton for the last eight years. The course is taken by 150–175 freshmen every spring who have taken one term of freshman mathematics including polynomial calculus during the fall term. The course is taken as a terminal course by students who go into the social and biological sciences and humanities. It has been taught effectively and with keen interest by instructors with special interests in all fields of pure and applied mathematics who were not familiar with the material at the outset. Unfortunately students of the physical sciences and engineering do not get exposed to these ideas at the

underclass level. Very few get exposed to them at any level. However, I believe the material could be very effectively taught if it were spread throughout a modernized freshman-sophomore mathematics sequence, the scheduling of the topics depending on the introduction and development of mathematical concepts and machinery. I believe it would not only improve the sequence for students of physics, chemistry and engineering, but it would make the course more effective for students who intend to go into mathematics and the social and biological sciences.

5. Summary and Conclusions. As we all know, widespread discussion of the underclass college mathematics curriculum has been going on for several years. Out of all of this discussion will come some freshman and sophomore courses which will undoubtedly look quite different from the traditional courses. Some of the directions of change are now quite evident. These directions of change look promising from the point of view of introducing the basic concepts of probability and statistics, now occurring so widely in all branches of science and technology. Good progress, it seems to me, has been made toward introducing fundamental material on the theory of sets and elementary probability. I am especially thinking about the three books I mentioned earlier in this paper. However, I would like to put in a plea for extending the material on elementary probability to include basic concepts of statistical inference more or less along the lines I have indicated. The main reasons for including this material in the main freshman-sophomore mathematics sequence can be summarized as follows:

1. The basic concepts of statistical inference occur as a natural and important pay-off extension of probability theory and are essentially mathematical concepts and can be effectively taught in a mathematics department.

2. The present system of teaching the basic concepts of statistical inference through elementary and largely nonmathematical courses at the upperclass or graduate level in various departments is inadequate. In the first place, the persons teaching these courses have, in general, only a secondary interest in statistics; they are usually primarily interested in some substantive area in their own department. Under this system the students do not get the basic statistical ideas with enough precision and generality; they are too much couched in the language and examples of a single discipline. In the second place, the students do not study the material early enough in their college careers. In many departments, including not only the social and biological sciences but the physical sciences and engineering, students need some of the basic concepts and methods of probability and statistics in their upperclass courses and experimental work.

3. These concepts occur in almost every field of social and natural science, engineering and technology, and should be a part of the general education of any student going into any of these areas. The only department in the colleges and universities of the country now sufficiently institutionalized to teach these basic concepts as broadly as they deserve to be taught is the mathematics de-

partment. While statistics departments have recently been organized in some of the large universities there are only a few of them and they are operating at present mainly at the upperclass and graduate levels. Eventually they will develop underclass programs. However, there are hundreds of colleges and universities in the country which do not have statistics departments, and most of the smaller ones may never have them. In the meantime, the best hope for getting these basic ideas taught effectively and to a sufficiently broad range of students in all of these colleges and universities is through their main freshman-sophomore sequence of mathematics courses. Even though relatively few mathematics teachers are now familiar with the basic concepts of statistical inference, I am confident that they will find them interesting and that they can pick them up quickly by teaching the material or by attending a summer institute such as those which have been sponsored by the National Science Foundation.

4. Teaching the basic ideas of probability and statistical inference in a modernized freshman-sophomore mathematics sequence will enable the various departments of a college or university now teaching statistics courses to concentrate on the nonintroductory probability and statistical problems and methods, including the nonmathematical aspects of statistical problems, more peculiar to the subject matter of those departments. This plan would assume a knowledge of the basic ideas of probability and statistics taught in such a freshman-sophomore mathematics sequence. The departments in a position to benefit would not only be economics, sociology, biology, and psychology but would include physics, chemistry, and engineering.

The probability material now in a modernized freshman-sophomore mathematics sequence plus the extension to statistical inference along the lines I have outlined could be organized, I think, so as to occupy between one-sixth and one-fifth of the time of a two-year sequence. This does not seem excessive for a body of mathematical ideas which, during the last thirty or forty years, have come to play a key role in nearly every field of science, technology and observational inquiry. The problem of effectively integrating these concepts into a new freshman-sophomore mathematics sequence is, of course, no easy task. But it deserves the most earnest consideration of all who are interested in a modern underclass mathematics curriculum.

THE MATHEMATICS OF SENTENCE STRUCTURE*

JOACHIM LAMBEK, McGill University

The definitions [of the parts of speech] *are very far from having attained the degree of exactitude found in Euclidean geometry.*

—Otto Jespersen, 1924.

1. Introduction. The aim of this paper is to obtain an effective rule (or algorithm) for distinguishing sentences from nonsentences, which works not only for the formal languages of interest to the mathematical logician, but also for natural languages such as English, or at least for fragments of such languages. An attempt to formulate such an algorithm is implicit in the work of Ajdukiewicz.† His method, later elaborated by Bar-Hillel [2], depends on a kind of arithmetization of the so-called *parts of speech*, here called *syntactic types*.‡

The present paper begins with a new exposition of the theory of syntactic types. It is addressed to mathematicians with at most an amateur interest in linguistics. The choice of sample languages is therefore restricted to English and mathematical logic. For the same reason, technical terms have been borrowed from the field of high school grammar.

Only a fragmentary treatment of English grammar is presented here. This should not be taken too seriously, but is meant to provide familiar illustrations for our general methods. The reader should not be surprised if he discovers considerable leakage across the line dividing sentences from nonsentences. It is only fair to warn him that some authorities think that such difficulties are inherent in the present methods.§ We take consolation in the words of Sapir: "All grammars leak."

The second part of this paper is concerned with a development of the technique of Ajdukiewicz and Bar-Hillel in a mathematical direction. We introduce a calculus of types, which is related to the well-known calculus of residuals.**

* This paper was written while the author held a Summer Research Associateship from the National Research Council of Canada. The present discussion of English grammar, in its final form, owes much to the careful reading and helpful criticism of earlier versions by Bar-Hillel and Chomsky.

† An English translation of his paper [1] is available in mimeographed form at the University of Chicago.

‡ Historically, these types can be traced back to the *semantic types* attributed by Tarski [21, p. 215] to E. Husserl and S. Lesniewski. A similar technique for logical systems was developed independently by Church [8]. Closely related is also the work by Curry [11] on *functional characters*. These correspond approximately to syntactic types for languages in which functors are always written on the left of their arguments.

§ Chomsky [6; 7] believes that such methods can describe only a small proportion of the sentences of a natural language and that other sentences should be obtained from these by certain transformations.

** See [3, XIII]. The calculus presented here is formally identical with a calculus constructed by G. D. Findlay and the present author for a discussion of canonical mappings in linear and multilinear algebra.

The decision problem for this system is solved affirmatively, following a procedure first proposed by Gentzen for the intuitionistic propositional calculus.††

The methods described here may be applied in several fields. For the teaching of English they provide a rigorous version of the traditional activity known as *diagramming* and *parsing*. For introductory logic courses they offer an effective definition of *well-formed formulas*. For the mechanical translation of languages [16], they may help with the syntactic analysis of the input material and indicate how to arrange the output into grammatical sentences of the target language. For the construction of an auxiliary language, they tell how to achieve a completely regular syntax; this is of special importance when the auxiliary is to act as an intermediate language in mechanical translation.

2. Syntactic types. While linguists are primarily interested in speech rather than in written texts, we shall here confine attention to the latter, if only to escape the difficult task of breaking up continuous discourse into discrete words. By a *word* we shall understand a word-form: Such forms as *work*, *works*, *worked* and *working* are different words; but the word *work* occurs twice in *we work best when we like our work*, although it functions as a verb in the first place and as a noun in the second. To describe the function of a word or expression we ascribe to it a certain *syntactic type*. This concept will now be defined; it corresponds approximately to the traditional *part of speech*.

We begin by introducing two *primitive* types: s, the type of sentences, and n, the type of names. For the sake of simplicity, we here restrict *sentence* to denote complete declarative sentences, ruling out requests and questions (as well as most replies, which are usually incomplete). By a *name* we understand primarily a proper name, such as *John* or *Napoleon*. But we shall also assign type n to all expressions which can occur in any context in which all proper names can occur. Thus type n is ascribed to the so-called class-nouns *milk*, *rice*, $\cdots$, which can occur without article, and to compound expressions such as *poor John*, *fresh milk*, $\cdots$.‡‡ We do not need to assign type n to the so-called count-nouns *king*, *chair*, $\cdots$, which require an article, nor to the pronoun *he*, as it cannot replace *John* in *poor John works* or *milk* in *John likes milk*.

From the primitive types we form compound types, by the recursive definition: If x and y are types, then so are x/y (read x *over* y) and $y\backslash x$ (read y *under* x). The meaning of these two kinds of division will be made clear by two examples.

The adjective *poor* modifies the name *John* from the left, producing the noun-phrase *poor John*. We assign to it type n/n.

The predicate (intransitive verb) *works* transforms the name *John* from the right into the sentence *John works*. We assign to it type $n\backslash s$.

In general, an expression of type x/y when followed by an expression of type

†† See [13; 10, II; 15, XV]. Curry [11, appendix] has also observed the close analogy between the theory of functional characters and the propositional calculus.

‡‡ There is a difficulty here: Of course we cannot check *all* admissible name contexts (whose number is infinite) to see whether *poor John* can be fitted in. Our assignment of types is tentative and subject to future revision.

y produces an expression of type x, and so does an expression of type $y\backslash x$ when preceded by an expression of type y. We write symbolically

(I) $$(x/y)y \to x, \qquad y(y\backslash x) \to x.$$

3. Type list for a fragment of English. We shall illustrate the assignment of types to English words by considering a number of sample sentences.

(1) *John works*

$n \quad n\backslash s$

This remains a sentence if *John* is replaced by any other name, hence *works* has type $n\backslash s$.

(2) *(poor John) works*

$n/n \quad n \quad n\backslash s$

Here *poor John* takes the place of the name in (1); in fact *poor John* can occur in any context in which all names can occur, hence it has type n. Moreover, so has *poor Tom*, *poor Jane*, $\cdots$, thus *poor* has type n/n.

(3) *(John works) here*

$n \quad n\backslash s \quad s\backslash s$

The word *here* transforms (1), or any other sentence, into a new sentence, hence it has type $s\backslash s$. The question may be raised whether *here* can be attached to a sentence such as (3) itself. While *John works here here* is open to stylistic objections, we shall consider it grammatically well-formed.

(4) *John (never works)*

$n \quad (n\backslash s)/(n\backslash s) \quad n\backslash s$

Since *John* can be replaced by any name here, *never works* has type $n\backslash s$, and so has *never sleeps*, $\cdots$; hence *never* has type $(n\backslash s)/(n\backslash s)$. It may be argued that (3) could also have been grouped *John* (*works here*) suggesting the type $(n\backslash s)\backslash(n\backslash s)$ for *here*. It will be shown later that every adverbial expression of type $s\backslash s$ also has type $(n\backslash s)\backslash(n\backslash s)$.

(5) *(John works) (for Jane)*

$n \quad n\backslash s \quad (s\backslash s)/n \quad n$

This indicates that *for Jane* should have the same type as *here* in (3), namely $s\backslash s$, and since *Jane* can be replaced by any other name *for* has type $(s\backslash s)/n$.

(6) *(John works) (and (Jane rests))*

$n \quad n\backslash s \quad (s\backslash s)/s \quad n \quad n\backslash s$

This illustrates how *and* can join two arbitrary sentences to form a new sentence; its type is therefore $(s\backslash s)/s$.

(7)
$$\begin{array}{ccc} John & (likes & Jane) \\ n & (n\backslash s)/n & n \end{array}$$

Here *likes Jane* has the same type as *works* in (1), hence *likes* has type $(n\backslash s)/n$. Similarly we may write *John* (*likes milk*) and even *milk* (*likes John*). The latter is a grammatical sentence, though open to semantic objections.

Example (7) raises an important point. Let us group this sentence

(7′)
$$\begin{array}{ccc} (John & likes) & Jane \\ n & n\backslash(s/n) & n \end{array}$$

Here *John likes* has type s/n, hence *likes* must be given the new type $n\backslash(s/n)$. We would regard the two types of *likes* in (7) and (7′) as in some sense equivalent. Abstracting from this particular situation, we write symbolically

(II)
$$(x\backslash y)/z \rightleftarrows x\backslash(y/z).$$

In practical applications it is often tedious to distinguish between equivalent types, we then write $x\backslash y/z$ for either side of (II). Further examples of this convention are afforded by the types of *never*, *for* and *and* [see Table I]. To avoid multiplication of parentheses, we may also abbreviate $(x/y)/z$ as $x/y/z$, and, symmetrically, $z\backslash(y\backslash x)$ as $z\backslash y\backslash x$. However, parentheses must not be omitted in such compounds as $x/(y/z)$, $(z\backslash y)\backslash x$, $(x/y)\backslash z$ and $z/(y\backslash x)$.

Table I compares the syntactic types of the words discussed above with the traditional parts of speech and the recent classification of Fries [12].

TABLE I

	Word	Type	Part of Speech	Fries Class
(1)	*works*	$n\backslash s$	intransitive verb	2C
(2)	*poor*	n/n	adjective	3
(3)	*here*	$s\backslash s$	adverb	4
(4)	*never*	$n\backslash s/(n\backslash s)$	adverb	
(5)	*for*	$s\backslash s/n$	preposition	F
(6)	*and*	$s\backslash s/s$	conjunction	E, J
(7)	*likes*	$n\backslash s/n$	transitive verb	2B

It is fairly clear that in this manner we can build up a type list for a gradually increasing portion of English vocabulary. This should be subject to possible revision, as more information becomes available.

To distinguish between different forms such as *works* and *work*, usually represented by a single dictionary entry, it is necessary to allow for more than two primitive types. Thus we might assign the type n^* to all noun-plurals, such as *men*, *chairs*, · · · . In contrast to examples (1), (2), (5), (7) we then have

(1*)
$$\begin{array}{cc} men & work \\ n^* & n^*\backslash s \end{array}$$

(2*) *poor men work*

n^*/n^* n^* $n^*\backslash s$

(5*) *John works for men*

n $n\backslash s$ $s\backslash s/n^*$ n^*

(7*) *John likes girls, men like Jane, men like girls*

n $n\backslash s/n^*$ n^* n^* $n^*\backslash s/n$ n n^* $n^*\backslash s/n^*$ n^*

This assignment successfully distinguishes between the forms *work* and *works*, *like* and *likes*, but it introduces an undesirable multiplicity of types for *poor*, *for*, *like*, and *likes*. While French distinguishes the forms *pauvre* and *pauvres*, English fails to make a corresponding distinction.

A more thorough analysis of the English verb phrase would compel us to introduce further primitive types for the infinitive and the two kinds of participles of intransitive verbs. This would lead to some revision of the type list embodied in Table I. While giving a more adequate treatment of English grammar, such a program would not directly serve the purpose of the present paper.

4. Formal systems. Suppose we have before us a string of words whose types are given. Then we can compute the type of the entire expression, provided its so-called *phrase structure* has been made visible by some device such as brackets. Consider for example

John (likes (fresh milk))

n $n\backslash s/n$ $\underbrace{n/n \quad n}_{n}$

$\underbrace{\qquad\qquad}_{n\backslash s}$

$\underbrace{\qquad\qquad\qquad\qquad}_{s}$

The indicated computation can also be written in one line:

$$n((n\backslash s/n)((n/n)n)) \to n((n\backslash s/n)n) \to n(n\backslash s) \to s.$$

In the formal languages studied by logicians, this process offers an effective test whether a given grouped string of symbols is a well-formed formula. For in these languages each word (usually consisting of a single sign) has just one preassigned type, and the use of brackets is obligatory. Let us call expressions with built-in brackets *formulas*; then formulas may be defined recursively: Each word is a formula, and if A and B are formulas, so is (AB).

Brackets are usually omitted when this can be done without introducing ambiguity. Brackets are regularly omitted in accordance with Rule (II). Thus logicians write

$$p \rightarrow q$$
$$s \; s\backslash s/s \; s$$

rather than

$$p \; (\rightarrow \; q)$$
$$s \; (s\backslash s)/s \; s$$

Allowance being made for this convention, the sentence structure of a formalized language is completely determined by its type list. A number of examples will illustrate this.

1. The propositional calculus, according to one of its formulations, possesses an infinite sequence of propositional variables of type s, and two signs for negation and implication of types s/s and $s\backslash s/s$ respectively.

The Polish school of logicians prefer to write all functors on the left of their arguments; it is well-known [18, IV] that all brackets can then be omitted without introducing ambiguity. The implication sign in the Polish notation is therefore of type $s/s/s$.

2. Boolean algebra, rather redundantly formulated, contains an infinite sequence of individual variables, as well as the signs 0 and 1, all of type n, an accent (for complementation) of type $n\backslash n$, cap and cup of type $n\backslash n/n$, equality and inclusion signs of type $n\backslash s/n$.

3. Quine's mathematical logic [17], into which we here introduce a special sign for universal quantification, contains an infinite sequence of individual variables of type n, and signs for joint denial, universal quantification and membership of types $s\backslash s/s$, $s/s/n$ and $n\backslash s/n$ respectively.

4. The calculus of lambda conversion due to Church, with a special sign of type $n/n/n$ for application [18, p. 111], contains also an infinite sequence of individual variables x_i $(i=1, 2, \cdots)$ of type n, together with a parallel sequence λx_i of type $n/n/n$.

5. The syntactic calculus to be introduced in this paper contains a number of symbols for primitive types of type n, three connectives $\cdot$, $\backslash$, $/$ of type $n\backslash n/n$, and the sign $\rightarrow$ of type $n\backslash s/n$.

In the interpretation of formal languages [21, XVIII, Section 4] one usually assumes that expressions of type s denote truth values, expressions of type n denote members of a given domain of individuals, and expressions of type x/y or $y\backslash x$ denote functions from the class of entities denoted by expressions of type y into the class of entities denoted by expressions of type x.

The above discussion of formal systems is somewhat oversimplified. Thus in Quine's formulation of mathematical logic, no special symbol is used for universal quantification, and in Church's formulation of the calculus of lambda con-

version the sign for application is not written. The syntactic description of these languages in terms of types would be more complicated without the special symbols introduced here. In some languages it is important to distinguish between constants and variables of apparently the same type [see, *e.g.*, 1]. A description in terms of two primitive types is then no longer adequate.

5. Type computations in English. Suppose we wish to compute the type of a string of English words, which are taken from a given type list. We cannot proceed quite as directly as in the formal systems discussed above, for two reasons, which we shall pause to discuss.

First, brackets do not usually occur in English texts, unless we regard punctuation as a half-hearted attempt to indicate grouping. Two ways of inserting brackets into an expression such as *the daughter of the woman whom he loved* may lead to essentially different syntactic resolutions, which may be accompanied by different meanings.

Secondly, English words usually possess more than one type. We have seen some examples of this in Section 3; others are easily found: The adverbial expression *today* has type s/s or $s\backslash s$, depending on whether it precedes or follows the sentence modified. The word *sound* may be a noun, an adjective, or a verb, either transitive or intransitive, depending on the context. Some "chameleon" words possess a type which is systematically ambiguous, allowing them to blend into many different contexts. Thus *only*, of type x/x, can probably modify expressions of any type x, and *and*, of type $x\backslash x/x$, will join together expressions of almost any type x to form a compound of the same type.

A mechanical procedure for analyzing English sentences would consist of four steps:

I. Insert brackets in all admissible ways.

II. To each word assign all types permitted by a given finite type list. (We ignore for the moment the difficulty arising from words which possess a potentially infinite number of types, as do the chameleons *and* and *only*).

III. For each grouping and type assignment compute the type of the total expression.

IV. Select that method of grouping and that type assignment which yields the desired type s.

A simple example, in which the problem of grouping does not arise, is

time	*flies*
n	n^*
$n^*\backslash s/n$	$n\backslash s$
$n^*\backslash s/n^*$	$n\backslash s/n$
	$n\backslash s/n^*$

Only the assignment

$$\begin{array}{cc} \textit{time} & \textit{flies} \\ n & n\backslash s \end{array}$$

produces a declarative sentence. This may be contrasted with

$$\begin{array}{cccccc} (\textit{spiders} & \textit{time} & \textit{flies}) & \textit{without} & \textit{clocks}, \\ n^* & n^*\backslash s/n^* & n^* & s\backslash s/n^* & n^* \end{array}$$

and

$$\begin{array}{cccccc} (\textit{TIME} & \textit{flies} & (10{,}000\ \textit{copies})) & \textit{to} & \textit{Montreal}. \\ n & n\backslash s/n^* & n^*/n^* & n^* & s\backslash s/n & n \end{array}$$

6. Pronouns. So far we have confined attention to the computation Rules (I) and (II). We have had one indication that other rules may play a role: the discussion of Example (4) suggests the rule

$$s\backslash s \rightarrow (n\backslash s)\backslash(n\backslash s).$$

To give a heuristic introduction for the consideration of further rules, we enter into a short discussion of English pronouns.

(8)
$$\begin{array}{cccccc} \textit{he} & \textit{works}, & \textit{he} & \textit{likes} & \textit{Jane} \\ s/(n\backslash s) & n\backslash s & s/(n\backslash s) & n\backslash s/n & n \end{array}$$

Since *he* transforms such expressions as *works*, *likes Jane*, $\cdots$, of type $n\backslash s$ into sentences, we assign to it type $s/(n\backslash s)$. We could of course enlarge the class of names to include pronouns, but then we should be hard put to explain why *poor he works* and *Jane likes he* are not sentences. At any rate, the assignment of type $s/(n\backslash s)$ to *he* is valid, irrespective of whether we regard pronouns as names. In fact, by the same argument, the name *John* also has type $s/(n\backslash s)$. This point will be discussed later.

(9)
$$\begin{array}{cccccc} \textit{that's} & \textit{him}, & \textit{Jane} & \textit{likes} & \textit{him}, \\ s/n & (s/n)\backslash s & n & n\backslash s/n & (s/n)\backslash s \\ & & \textit{Jane} & \textit{works} & \textit{for} & \textit{him} \\ & & n & n\backslash s & s\backslash s/n & (s/n)\backslash s \end{array}$$

The expressions *that's*, *Jane likes* and *Jane works for* all have type s/n, hence we have ascribed type $(s/n)\backslash s$ to *him*. (This assignment is not quite correct:* The example *Jane likes poor John* indicates that the expression *Jane likes poor* also has type s/n, yet *Jane likes poor him* is not a sentence. Moreover the present assignment does not explain why *that's he* is a sentence in the speech of some people. We shall overlook these defects here.) We observe that the difference

* This was kindly pointed out to the author by N. Chomsky.

in form between *he* and *him* is reflected by a difference in type, indicating that the former operates from the left, while the latter operates from the right. Sapir [19, VII] has called these two forms the *pre-verbal* and *post-verbal* case of the pronoun respectively.

A difficulty arises when we try to show the sentencehood of

(10)
$$\begin{array}{ccccc} he & likes & him & ; \\ s/(n\backslash s) & n\backslash s/n & (s/n)\backslash s & \end{array}$$

for

$$(s/(n\backslash s))(n\backslash s/n)((s/n)\backslash s)$$

cannot be simplified any further by the Rules (I) and (II). We introduce two new rules

(III) $$(x/y)(y/z) \rightarrow x/z, \qquad (x\backslash y)(y\backslash z) \rightarrow x\backslash z.$$

We may then assign type

$$(s/(n\backslash s))(n\backslash s/n) \rightarrow s/n$$

to *he likes* and type

$$(n\backslash s/n)((s/n)\backslash s) \rightarrow n\backslash s$$

to *likes him*, permitting two equivalent resolutions

$$\begin{array}{ccccc} \underbrace{(he\ likes)} & him, & he & \underbrace{(likes\ him).} \\ s/n & (s/n)\backslash s & s/(n\backslash s) & n\backslash s \end{array}$$

Rules (III) also allow alternative, though equivalent, resolutions of expressions considered earlier; *e.g.*, the sentence

$$\begin{array}{cccc} (John\ works) & for & Jane \\ n & n\backslash s & s\backslash s/n & n \end{array}$$

can now also be grouped *John* (*works* (*for Jane*)), where the predicate has type

$$(n\backslash s)((s\backslash s/n)n) \rightarrow (n\backslash s)(s\backslash s) \rightarrow n\backslash s.$$

We have seen above that the name *John* also has the type of the pronoun *he*. For the same reason, it also has the type of the pronoun *him*. We symbolize the situation by writing

$$n \rightarrow s/(n\backslash s), \qquad n \rightarrow (s/n)\backslash s$$

and more generally

(IV) $$x \rightarrow y/(x\backslash y), \qquad x \rightarrow (y/x)\backslash x.$$

These new rules may actually be required for computations. Suppose that

from sample sentences such as *books by him bore* we arrived at the type $n^*\backslash s/n'$ for *by*, where n' is short for $(s/n)\backslash s$. The phrase *books by John* then requires the computation

$$n^*(n^*\backslash s/n')n \to (s/n')n \to (s/n')n' \to s$$

which utilizes rules (I), (IV) and (I) in this order.

While Ajdukiewicz [1] makes use of (III), Rules (IV) suggest that the mathematical apparatus used hitherto may have to be expanded.

7. Syntactic calculus. By an *expression* we shall mean a string of words. Let us suppose that to certain expressions there have been assigned certain primitive types. If A has type x and B has type y, we assign to the expression AB the type xy, also written $x \cdot y$. We assign type z/y to all expressions A such that AB has type z for any B of type y. We assign type $x\backslash z$ to all expressions B such that AB has type z for any A of type x. We write $x \to y$ to mean that any expression of type x also has type y. We write $x \rightleftarrows y$ to mean that $x \to y$ and $y \to x$.

The following rules are now valid:

(a) $x \to x$

(b) $(xy)z \to x(yz)$ (b′) $x(yz) \to (xy)z$

(c) *if* $xy \to z$ *then* $x \to z/y$ (c′) *if* $xy \to z$ *then* $y \to x\backslash z$

(d) *if* $x \to z/y$ *then* $xy \to z$ (d′) *if* $y \to x\backslash z$ *then* $xy \to z$

(e) *if* $x \to y$ *and* $y \to z$ *then* $x \to z$

Rules (a), (b), (b′), (e) hold trivially. Rules (c′) and (d′) are symmetric duals of (c) and (d), hence it suffices to prove the latter.

Assume $xy \to z$, and let A have type x. Then for any B of type y, AB has type z; hence A has type z/y. Thus $x \to z/y$.

Conversely, assume $x \to z/y$, and let A, B have types x, y, respectively; then AB has type z. Thus $xy \to z$.

The system presented above may be viewed abstractly as a formal language with a number of primitive type symbols of type n, three connectives $\cdot$, /, $\backslash$ of type $n\backslash n/n$, and a relation symbol $\to$ of type $n\backslash s/n$. If we furthermore regard (a), (b) and (b′) as axiom schemes and (c) to (e) as rules of inference, we obtain a deductive system which may be called *syntactic calculus*. A number of rules are provable in the system; for example,

(f) $x \to (xy)/y$,

(g) $(z/y)y \to z$,

(h) $y \to (z/y)\backslash z$,

(i) $(z/y)(y/x) \to z/x$,

(j) $z/y \to (z/x)/(y/x)$,

(k) $(x\backslash y)/z \rightleftarrows x\backslash(y/z)$,

(l) $(x/y)/z \rightleftarrows x/(zy)$,

(m) *if* $x \to x'$ *and* $y \to y'$ *then* $xy \to x'y'$,

(n) *if* $x \to x'$ *and* $y \to y'$ *then* $x/y' \to x'/y$.

Here (f) follows from $xy \to xy$ by (c), (g) follows from $z/y \to z/y$ by (d), (h) follows from (g) by (c'), (j) follows from (i) by (c). Proofs of (i), (k) and (l) are a bit longer; we omit them in view of the decision procedure established in Section 8. Proofs of (m) and (n) are arranged in tree form.

Proof of (m).

$$\dfrac{\dfrac{x \to x' \quad \dfrac{x'y \to x'y}{x' \to (x'y)/y}\,(c)}{x \to (x'y)/y}\,(e)}{xy \to x'y}\,(d) \qquad \dfrac{y \to y' \quad \dfrac{x'y' \to x'y'}{y' \to x'\backslash(x'y')}\,(c')}{x'y \to x'y'}\,(e)\,(d')$$

$$xy \to x'y' \qquad (e)$$

Proof of (n).

$$\dfrac{\dfrac{y \to y' \quad \dfrac{\dfrac{x/y' \to x/y'}{(x/y')y' \to x}\,(d)}{y' \to (x/y')\backslash x}\,(c')}{y \to (x/y')\backslash x}\,(e)}{(x/y')y \to x}\,(d') \qquad \dfrac{\dfrac{x/y \to x/y}{(x/y)y \to x}\,(d) \quad x \to x'}{(x/y)y \to x'}\,(e)$$

$$\dfrac{x/y' \to x/y}{x/y' \to x'/y}\,(c) \qquad (d)$$

$$x/y' \to x'/y \qquad (e)$$

The syntactic theorems (g), (h), (i), and (k) coincide with the Rules (I), (IV), (III), and (II), respectively. An illustration of (j), or rather its symmetric dual, appeared in Section 3, where it was pointed out that every sentence-modifying adverb is also a predicate-modifying adverb, symbolically,

$$s\backslash s \to (n\backslash s)\backslash(n\backslash s).$$

Rule (l) is due to Schönfinkel [20], who observed that a function of two variables may be regarded as an ordinary function of one variable whose value is again an ordinary function, so that

$$f(a, b) = (fa)b.$$

If a, b and $f(a, b)$ have types x, y and z respectively, then f occurs in $f(a, b)$ with type $z/(xy)$ and in $(fa)b$ with type $(z/y)/x$, these two types being equivalent by (l).

8. Decision procedure. Is there an effective method for testing whether a sentence $x \rightarrow y$ of the syntactic calculus is deducible from rules (a) to (e)? This is the so-called decision problem for the syntactic calculus. It turns out that the decision procedure discovered by Gentzen [**15**, XV] for the intuitionistic propositional calculus can be adapted for the present purpose.

Following Gentzen, we define the *sequent*

$$x_1, x_2, \cdots x_n \rightarrow y$$

to stand for

$$(\cdots((x_1x_2)x_3)\cdots x_n) \rightarrow y,$$

where $x_1, \cdots, x_n, y$ are types. Now let x be any of the possible products of the x_i obtained from some way of grouping the string $x_1x_2 \cdots x_n$. Then it follows by repeated application of rules (b), (b′), (m) and (e) that

$$x \rightleftarrows (\cdots((x_1x_2)x_3)\cdots x_n).$$

Hence the above sequent is also equivalent to the formula $x \rightarrow y$.

Let capitals denote sequences of types, possibly empty sequences. By "U, V" we mean the sequence obtained by juxtaposing U and V; if U is empty it means V, and if V is empty it means U. The following rules are consequences of (a) to (e), provided T, P and Q are not empty.

(1) $x \rightarrow x$

(2) *if* $T, y \rightarrow x$ *then* $T \rightarrow x/y$

(2′) *if* $y, T \rightarrow x$ *then* $T \rightarrow y\backslash x$

(3) *if* $T \rightarrow y$ *and* $U, x, V \rightarrow z$ *then* $U, x/y, T, V \rightarrow z$

(3′) *if* $T \rightarrow y$ *and* $U, x, V \rightarrow z$ *then* $U, T, y\backslash x, V \rightarrow z$

(4) *if* $U, x, y, V \rightarrow z$ *then* $U, xy, V \rightarrow z$

(5) *if* $P \rightarrow x$ *and* $Q \rightarrow y$ *then* $P, Q \rightarrow xy$

Note that each of Rules (2) to (5) introduces an occurrence of one of the connectives $\cdot$, /, \ into the conclusion.

To derive Rules (1) to (5) from (a) to (e), we observe that (1) is the same as (a), (2) becomes (c), (2′) becomes (c′), (4) is immediate, and (5) becomes (m), if the sequences T, U, V, P, and Q are replaced by the products of the terms in them. It remains only to prove (3), since (3′) is its symmetric dual.

First let us take the case where U and V are empty sequences. We replace T by some product t of its terms. Then (3) takes the form: *if* $t \to y$ *and* $x \to z$ *then* $(x/y)t \to z$. This may be shown thus:

$$\dfrac{\dfrac{x \to z \quad t \to y}{x/y \to z/t}\,(\mathrm{n})}{(x/y)t \to z}\,(\mathrm{d})$$

Next suppose U is empty but V is not. Replace the latter by a product v of its terms. Then (3) takes the form: *if* $t \to y$ *and* $xv \to z$ *then* $((x/y)t)v \to z$. This is established thus:

$$\dfrac{\dfrac{\dfrac{xv \to z}{x \to z/v}\,(\mathrm{c}) \quad t \to y}{(x/y)t \to z/v}\,(\text{as above})}{((x/y)t)v \to z}\,(\mathrm{d})$$

Similarly we deal with the remaining two cases in which U is not empty.

Conversely, we shall deduce rules (a) to (e) from (1) to (5), so that the two sets of rules are equivalent. For the moment we assume one additional rule, the so-called *cut*,

(6) *if* $T \to x$ *and* $U, x, V \to y$ *then* $U, T, V \to y$

It will appear later (Gentzen's theorem) that this new rule does not increase the set of theorems deducible from (1) to (5).

Now (a) coincides with (1), and (e) is a special case of (6), hence it suffices to prove (b), (c) and (d). Proofs are arranged in tree form.

Proof of (b).

$$\dfrac{\dfrac{\dfrac{\dfrac{y \to y \quad z \to z}{x \to x \quad y, z \to yz}\,(5)}{x, y, z \to x(yz)}\,(5)}{xy, z \to x(yz)}\,(4)}{(xy)z \to x(yz)}\,(4)$$

Proof of (c).

$$\dfrac{\dfrac{\dfrac{x \to x \quad y \to y}{x, y \to xy}\,(5)}{x \to (xy)/y}\,(2) \qquad \dfrac{y \to y \quad xy \to z}{(xy)/y, y \to z}\,(3)}{x, y \to z}\,(6)$$
$$\dfrac{}{x \to z/y}\,(2)$$

Proof of (d).

$$\dfrac{\dfrac{x \to z/v \quad v \to v}{x, v \to (z/v)v}\,(5) \qquad \dfrac{\dfrac{v \to y \quad z \to z}{z/y, v \to z}\,(3)}{(z/v)y \to z}\,(4)}{\dfrac{x, v \to z}{xy \to z}\,(4)}\,(6)$$

Let us verify that we have, in fact, a decision procedure. Given a sequent $U \to x$, we attempt to construct a proof in tree form, working from the bottom up, using Rules (1) to (5), but not (6). Every upward step eliminates an occurrence of one of the connectives $\cdot$, /, \, and there are only a finite number of ways of making this step. Therefore the total number of proofs that can be attempted is finite. The sequent $U \to x$ is deducible if and only if one of the attempted proofs is successful.

9. Proof of Gentzen's theorem. If $T \to x$ and $U, x, V \to y$ are both provable according to Rules (1) to (5), we will show that $U, T, V \to y$ is also provable, so that we may adopt as a new rule of inference the cut

$$\dfrac{T \to x \quad U, x, V \to v}{U, T, V \to y}\,(6).$$

We prove this by reduction on the *degree* of the cut, which is defined thus: Let $d(x)$ be the number of separate occurrences of the connectives $\cdot$, /, \ in the type formula x, and let

$$d(x_1, x_2, \cdots x_n) = d(x_1) + d(x_2) + \cdots d(x_n),$$

then the degree of the above cut is

$$d(T) + d(U) + d(V) + d(x) + d(y).$$

We will now show that in any cut, whose premises have been proved without cut, the conclusion is either identical with one of the premises, or else the cut can be replaced by one or two such cuts of smaller degree. Since no degree is negative, this will establish Gentzen's theorem. We consider seven cases, which need not be mutually exclusive.

Case 1. $T \to x$ is an instance of (1); then $T = x$ and the conclusion coincides with the other premise.

Case 2. $U, x, V \to y$ is an instance of (1); then U and V are empty and $x = y$. Hence the conclusion coincides with the premise $T \to x$.

Case 3. The last step in the proof of $T \to x$ uses one of Rules (2) to (5), but does not introduce the main connective of x. Then $T \to x$ is inferred by Rule (3), (3′) or (4) from one or two sequents, one of which has the form $T' \to x$ with

$d(T') < d(T)$. The cut

$$\frac{T' \to x \quad U, x, V \to y}{U, T', V \to y}\,(6)$$

has smaller degree than the given cut. Moreover the rule which led from $T' \to x$ to $T \to x$ will also lead from $U, T', V \to x$ to $U, T, V \to x$, as may be easily verified in the different subcases.

Case 4. The last step in the proof of $U, x, V \to y$ uses one of Rules (2) to (5), but does not introduce the main connective of x. Then $U, x, V \to y$ is inferred from one or two sequents, one of which has the form $U', x, V' \to y'$. Since the inference introduces an occurrence of one connective,

$$d(U') + d(V') + d(y') < d(U) + d(V) + d(y).$$

Therefore the cut

$$\frac{T \to x \quad U', x, V' \to y'}{U', T, V' \to y'}\,(6)$$

has smaller degree than the given cut. Moreover, the same rule which led from $U', x, V' \to y'$ to $U, x, V \to y$ will lead from $U', T, V' \to y'$ to $U, T, V \to y$, as is easily verified in the different subcases.

Case 5. The last steps in the proofs of both premises introduce the main connective of $x = x'x'' = x' \cdot x''$. We may replace

$$\cfrac{\cfrac{T' \to x' \quad T'' \to x''}{T', T'' \to x'x''}\,(5) \qquad \cfrac{U, x', x'', V \to y}{U, x'x'', V \to y}\,(4)}{U, T', T'', V \to y}\,(6)$$

by

$$\cfrac{T'' \to x'' \qquad \cfrac{T' \to x' \quad U, x', x'', V \to y}{U, T', x'', V \to y}\,(6)}{U, T', T'', V \to y}\,(6)$$

where both new cuts have smaller degree.

Case 6. The last steps in the proofs of both premises introduce the main connective of $x = x'/x''$. We may replace

$$\cfrac{\cfrac{T, x'' \to x'}{T \to x'/x''}\,(2) \qquad \cfrac{V' \to x'' \quad U, x', V'' \to y}{U, x'/x'', V', V'' \to y}\,(3)}{U, T, V', V'' \to y}\,(6)$$

by

$$\dfrac{\dfrac{T, x'' \to x' \quad U, x', V'' \to y}{V' \to x'' \quad U, T, x'', V'' \to y}\,(6)}{U, T, V', V'' \to y}\,(6)$$

where both new cuts have smaller degree.

Case 7. This last case is like Case 6, except that $x = x''\backslash x'$, and is treated symmetrically.

10. Algebraic remarks. The following remarks may be of mathematical interest. If we write $=$ instead of $\rightleftarrows$, the deductive system studied here becomes a partially ordered system which resembles a residuated lattice [3, XIII]. It may be mapped homomorphically onto a free group by mapping each element x onto its congruence class modulo $\equiv$, where $x \equiv y$ means that there exists a sequence $x = x_1, \cdots, x_n = y$ $(n \geqq 1)$, such that $x_i \to x_{i+1}$ or $x_{i+1} \to x_i$ $(1 \leqq i < n)$. If this group is abelianized, we obtain something very much like the group of dimension symbols, which plays an important role in the grammar of physics.

It turns out that $x \equiv y$ if and only if

(1) $$x \to t \text{ and } y \to t \text{ for some } t,$$

or equivalently

(2) $$z \to x \text{ and } z \to y \text{ for some } z.$$

This result is easily proved by induction on the length of the given sequence connecting x with y, once the equivalence of (1) and (2) has been established.

Assuming (1), we put

$$z = (x/((t/t)\backslash t))((t/t)\backslash y)$$

and verify (2) by computation; assuming (2), we put

$$t = (x(z\backslash z))/(y\backslash(z(z\backslash z)))$$

and verify (1) by computation.

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VARIABLE MATRIX SUBSTITUTION IN ALGEBRAIC CRYPTOGRAPHY

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1. Introduction. The use of algebraic methods in cryptography is well-known through two important papers by Hill [1], [2]. Briefly, the basic idea can be formulated in the following way. Consider the system of simultaneous congruences

$$y_i = \sum_{j=1}^{n} a_{ij}x_j \pmod{26}, \qquad i = 1, \cdots, n, \tag{1.1}$$

where the constants a_{ij} are chosen so that the determinant $|a_{ij}|$ is *prime* to 26. By means of (1.1) the set of n variables $(x_1, \cdots, x_n)$ is transformed to the set $(y_1, \cdots, y_n)$ and, conversely, the set $(y_1, \cdots, y_n)$ will be transformed to the unique set $(x_1, \cdots, x_n)$ by means of the inverse transformation which exists by the assumption on $|a_{ij}|$.

To each of the 26 letters of the alphabet we associate an integer from the set $0, 1, \cdots, 25$, so that no two letters correspond to the same integer. For simplicity we illustrate with the correspondence (used throughout this paper)

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
(1.2)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	0

Now to encipher a message, or plain text, by means of (1.1), first replace each letter of the text by means of its numerical equivalent, using for illustration, (1.2). Then divide the resulting sequence of numbers into groups containing n numbers each. Call these

$$p_{11}p_{12} \cdots p_{1n} \quad p_{21}p_{22} \cdots p_{2n} \quad \cdots \quad p_{i1}p_{i2} \cdots p_{in} \quad \cdots . \tag{1.3}$$

Each group of (1.3) is then used in (1.1) for $x_1 \cdots x_n$, and the transformed set

$y_1 \cdots y_n$ calculated. Call the sequence of these sets

$$c_{11}c_{12}\cdots c_{1n} \quad c_{21}c_{22}\cdots c_{2n} \quad \cdots \quad c_{i1}c_{i2}\cdots c_{in} \quad \cdots . \tag{1.4}$$

Convert the numbers of (1.4) into their letter equivalents by (1.2). These letters will constitute the cipher text corresponding to the given plain text.

The decipherment is accomplished by means of the inverse to (1.1).

As a concrete example, select $n=3$, and (1.1) as

$$\begin{aligned} y_1 &\equiv x_1 + 2x_2 + 3x_3 \\ y_2 &\equiv 2x_1 + 5x_2 + 6x_3 \\ y_3 &\equiv x_1 + 2x_2 + 4x_3 \end{aligned} \tag{1.5}$$

To encipher the text CRYPTOGRAPHIC, divide into sets of three letters, adding, say, XX to complete the last set:

C	R	Y	P	T	O	G	R	A	P	H	I	C	X	X
3	18	25	16	20	15	7	18	1	16	8	9	3	24	24

The sequence (1.3) is here 3 18 25 16 20 15 $\cdots$. Substitute the first set 3 18 25 ($=p_{11}p_{12}p_{13}$) in (1.5) for $x_1x_2x_3$ to give

$$\begin{aligned} y_1 &\equiv 3 + 36 + 75 = 114 \equiv 10 = \text{J}, \\ y_2 &\equiv 6 + 90 + 150 = 246 \equiv 12 = \text{L}, \\ y_3 &\equiv 3 + 36 + 100 = 139 \equiv 9 = \text{I}, \end{aligned}$$

Here the first cipher sequence $c_{11}c_{12}c_{13}$ of (1.4) is 10 12 9, which converted to letters by (1.2) gives JLI as shown.

The complete encipherment proceeds as above, and produces

JLI WNL TFU GVP SJQ

To decipher, obtain the inverse of (1.5),

$$\begin{aligned} x_1 &\equiv 8y_1 + 24y_2 + 23y_3 \\ x_2 &\equiv 24y_1 + y_2 \\ x_3 &\equiv 25y_1 + y_3 \end{aligned} \tag{1.6}$$

(The congruences are of course taken mod 26, in which $25=-1$, $24=-2$, *etc.*) The reciprocal of a prime p, mod 26, is q, where $pq \equiv 1 \bmod 26$.

Now using JLI = 10 12 9 as $y_1y_2y_3$ in (1.6) gives

$$\begin{aligned} x_1 &= 80 + 288 + 207 = 575 \equiv 3 = \text{C} \\ x_2 &\equiv 240 + 12 = 252 \equiv 18 = \text{R} \\ x_3 &\equiv 250 + 9 = 259 = 25 = \text{Y} \end{aligned}$$

or CRY, the first plain-text group. The rest of the plain text is found in like manner. (In actual practice we would use -1 for 25, -2 for 24, *etc.*, in (1.6).)

In Hill's papers the transformation (1.1) is generalized by the use of matrix coefficients, but the above is sufficient for our purpose.

(The author notes in passing that simultaneous equations were used by him for cryptographic purposes to a limited extent several years prior to the appearance of Hill's papers.)

2. Fixed substitution. The cryptographic method represented by (1.1) is known as a *fixed substitution system*. This means that any given plain-text group will always be replaced by the same cipher-text group. This is true because the coefficients a_{ij} remain fixed throughout the encipherment of a message.

From a cryptographic point of view there is a distinct advantage in using a *variable substitution* method, whereby the various appearances of a given plain-text group will be replaced by different cipher groups. It is our purpose to indicate some simple ways to accomplish this based on (1.1).

3. Variable substitution, first method. It is convenient to represent (1.1) as a matrix congruence

$$C \equiv AP \pmod{26}, \tag{3.1}$$

where matrices A, C, P are defined by

$$A = \begin{bmatrix} a_{11} \cdots a_{1n} \\ \cdot \quad\quad \cdot \\ \cdot \quad\quad \cdot \\ a_{n1} \cdots a_{nn} \end{bmatrix}, \qquad P = \begin{bmatrix} p_1 \\ \cdot \\ \cdot \\ p_n \end{bmatrix}, \qquad C = \begin{bmatrix} c_1 \\ \cdot \\ \cdot \\ c_n \end{bmatrix} \tag{3.2}$$

and P, C are one-column matrices representing corresponding plain- and cipher-text groups.

Now in classical cryptography several variable substitution methods are well-known. These can be represented by the congruences

$$c_i \equiv p_i + k_i \pmod{26}, \qquad i = 1, \cdots, N, \tag{3.3}$$

where p_i is the numerical value of the ith plain-text letter according to some correspondence as (1.2), c_i is the numerical value of the corresponding cipher-text letter, N is number of letters in the message, and the sequence of numbers $k_1k_2 \cdots$ has one of the following properties:

(a) $k_1k_2 \cdots$ is a periodic sequence, say $k_1k_2 \cdots k_mk_1k_2 \cdots k_mk_1 \cdots$, where the numbers $k_1 \cdots k_m$ of the period are selected in any preassigned manner.

(b) The number k_i is chosen by the relation

$$k_i = c_{i-1} \tag{3.4}$$

so

$$c_i \equiv p_i + c_{i-1} \qquad (c_0 \text{ chosen in advance}). \tag{3.5}$$

(c) The number k_i is chosen by the relation

$$k_i = p_{i-1}, \tag{3.6}$$

so

$$c_i \equiv p_i + p_{i-1} \qquad (p_0 \text{ chosen in advance}). \tag{3.7}$$

Note that in each of the above three methods p_i is uniquely determined in the decipherment process. This is, of course, a prime requisite in any cryptographic system.

In matrix form (3.3) can be written as

$$C \equiv P + K \tag{3.8}$$

where the indicated matrices are each of one row and one column, since (3.3) represents encipherment one letter at a time.

Now to obtain a variable substitution analogous to (3.1) we generalize (3.8) to

$$C \equiv AP + BK \pmod{26}, \tag{3.9}$$

where $A = [a_{ij}]$, $B = [b_{ij}]$ are $n \times n$ matrices with fixed elements (and $|A|$ prime to 26). Matrices C and P are as given in (3.2), and K is a one-column matrix,

$$K = \begin{bmatrix} k_1 \\ \cdot \\ \cdot \\ \cdot \\ k_n \end{bmatrix}.$$

Corresponding to the three cases (a), (b), (c) above for choosing the k_i, we have:

(a′) Define matrices

$$C_i = \begin{bmatrix} c_{i1} \\ \cdot \\ \cdot \\ \cdot \\ c_{in} \end{bmatrix}, \qquad P_i = \begin{bmatrix} p_{i1} \\ \cdot \\ \cdot \\ \cdot \\ p_{in} \end{bmatrix}, \qquad K_i = \begin{bmatrix} k_{i1} \\ \cdot \\ \cdot \\ \cdot \\ k_{in} \end{bmatrix}, \tag{3.10}$$

using (1.3), (1.4), and

$$K_i = K_{i+m}, \qquad i = 1, 2, \cdots, \tag{3.11}$$

where $K_1, \cdots, K_m$ are chosen in any preassigned manner.

Then

$$C_i \equiv AP_i + BK_i, \qquad i = 1, 2, \cdots, \tag{3.12}$$

from (3.9) gives the substitution. Also,

$$P_i \equiv A^{-1}C_i - A^{-1}BK_i. \tag{3.13}$$

(b′) In this case we choose $K_i = C_{i-1}$, so

$$C_i \equiv AP_i + BC_{i-1} \qquad (C_0 \text{ chosen in advance}).$$

(c′) Choose $K_i = P_{i-1}$, so

$$C_i \equiv AP_i + BP_{i-1} \qquad (P_0 \text{ chosen in advance}). \tag{3.14}$$

To obtain involutory transformations (in which a transformation and its inverse are identical) we have from (3.12), (3.13),

$$A = A^{-1}, \qquad B = -A^{-1}B = -AB, \tag{3.15}$$

and (3.13) becomes $P_i = AC_i + BK_i$. A solution of (3.15) is

$$A^2 = I, \qquad B = A - I \qquad (I = \text{identity matrix}).$$

To obtain A such that $A^2 = I$, a formula in [2] may be used,

$$a_{ij} \equiv \delta_{ij} - \tau\lambda_i\lambda_j, \quad \sigma\tau \equiv 2 \pmod{26}, \quad \sigma \equiv \sum_1^n \lambda_i^2 \pmod{26}, \tag{3.16}$$

σ must be prime to 26.

We illustrate case (c′) using

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 1 & 2 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & 1 & 1 \\ 2 & 0 & 3 \\ 1 & 2 & 0 \end{bmatrix}, \qquad P_0 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},$$

$$\begin{bmatrix} c_{i1} \\ c_{i2} \\ c_{i3} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} p_{i1} \\ p_{i2} \\ p_{i3} \end{bmatrix} + \begin{bmatrix} 4 & 1 & 1 \\ 2 & 0 & 3 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} p_{i-1,1} \\ p_{i-1,2} \\ p_{i-1,3} \end{bmatrix}. \tag{3.17}$$

To encipher CRYPTOGRAPHIC, we have

$$\begin{bmatrix} c_{11} \\ c_{12} \\ c_{13} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 18 \\ 25 \end{bmatrix} + \begin{bmatrix} 4 & 1 & 1 \\ 2 & 0 & 3 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 19 \\ 23 \\ 14 \end{bmatrix} = \begin{bmatrix} \text{S} \\ \text{W} \\ \text{N} \end{bmatrix},$$

$$\begin{bmatrix} c_{21} \\ c_{22} \\ c_{23} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 16 \\ 20 \\ 15 \end{bmatrix} + \begin{bmatrix} 4 & 1 & 1 \\ 2 & 0 & 3 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 18 \\ 25 \end{bmatrix} = \begin{bmatrix} 0 \\ 17 \\ 25 \end{bmatrix} = \begin{bmatrix} \text{Z} \\ \text{Q} \\ \text{Y} \end{bmatrix},$$

$$\begin{bmatrix} c_{31} \\ c_{32} \\ c_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 7 \\ 18 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 & 1 & 1 \\ 2 & 0 & 3 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 16 \\ 20 \\ 15 \end{bmatrix} = \begin{bmatrix} 15 \\ 5 \\ 25 \end{bmatrix} = \begin{bmatrix} \text{O} \\ \text{E} \\ \text{Y} \end{bmatrix}, \textit{ etc.}$$

The complete encipherment is SWN ZQY OEY BMG VQW, using CXX as the last plain group.

The decipherment can be obtained from the inverse to (3.17),

$$P_i = \begin{bmatrix} 8 & 24 & 23 \\ 24 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix} C_i + \begin{bmatrix} 1 & 24 & 24 \\ 6 & 2 & 25 \\ 3 & 25 & 1 \end{bmatrix} P_{i-1}.$$

4. Variable substitution, second method. We return to the basic relation (3.1) and attempt to replace the matrix A of fixed elements by a matrix with variable elements. A general situation is obtained if the elements a_{ij} of A be considered as polynomial functions of a set of parameters $t, u, v, \cdots$ in such a way that the determinant $|A|$ of A is *independent of the parameters* and is a prime number mod 26. The inverse A^{-1} of A will then exist for all parameter values, and hence $P = A^{-1}C$ can always be found. We consider one of the simpler cases here.

Any triangular matrix

$$T = \begin{bmatrix} t_{11} & 0 & \cdots & 0 \\ t_{21} & t_{22} & \cdots & \\ \cdot & \cdot & \cdots & \\ t_{n1} & t_{n2} & \cdots & t_{nn} \end{bmatrix}$$

with $t_{ij}(t, u, v \cdots)$, $(i \neq j)$, such that t_{ii} is a prime mod 26 will have for determinant $|T| = t_{11}t_{22} \cdots t_{nn}$, a prime mod 26. If T be transformed by elementary transformations leaving $|T|$ unchanged we can obtain a general matrix A of the desired property.

For example, using

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ t & 3 & 0 & 0 \\ 2t+1 & 2 & 5 & 0 \\ 1 & t & t+1 & 7 \end{bmatrix}, \qquad |T| \equiv 1, \text{ mod } 26,$$

we can transform T to

$$A(t) \equiv \begin{bmatrix} 1 & 1 & t & 2t \\ t & t+3 & t & 2t^2 \\ 2t+1 & 2t+3 & 2t^2+t+5 & 2t^2+2t \\ 1 & t+1 & 2t+1 & 2t+7 \end{bmatrix}, \qquad |A(t)| \equiv 1, \text{ mod } 26.$$

Place $C = A(t)P$. For each $P = P_i$ give t a value k_i determined in some preassigned manner,

$$C_i = A(k_i)P_i, \qquad P_i = A^{-1}(k_i)C_i. \tag{4.1}$$

Any of the methods (a′), (b′), (c′) can be used, taking for example,

$$k_i = c_{i-1,1} \text{ or } k_i = p_{i-1,1}, \text{ or } k_i = p_{i-1,1} + p_{i-1,2}, \text{ etc.},$$

in the latter two cases.

One disadvantage of this procedure to obtain $A(t)$ is that the elements of $A^{-1}(t)$ will in general be high degree polynomials in the parameters, thus causing computational difficulties.

One way to avoid this difficulty is to assume $A(t)$ is linear in t and impose the condition that $A^{-1}(t)$ is likewise. Thus, place

$$A(t) = G + tH, \tag{4.2}$$

with G, H constant element matrices, and $|G|$ a prime mod 26. It is easily shown $A^{-1}(t)$ will be linear in t if $H=XG$, $X^2=0$. Then

$$A^{-1}(t) = G^{-1} - tG^{-1}X. \tag{4.3}$$

To obtain a general matrix X satisfying $X^2=0$, define N by

$$N = \begin{bmatrix} 0 & n_1 & & & & & & & \\ 0 & 0 & & & & & & & \\ & & 0 & n_2 & & & & & \\ & & 0 & 0 & & & & & \\ & & & & \cdot & & & & \\ & & & & & \cdot & & & \\ & & & & & & 0 & n_q & \\ & & & & & & 0 & 0 & \\ & & & & & & & & 0 \\ & & & & & & & & \cdot \\ & & & & & & & & \cdot \\ & & & & & & & & 0 \end{bmatrix}, \qquad q \leqq [n/2]. \tag{4.4}$$

N consisting of all zeros except $n_1, n_2, \cdots, n_q$ placed immediately to the right of the main diagonal terms in alternate rows as shown. The n_i are arbitrary constants. It is evident that $N^2=0$. X is now defined by

$$X = QNQ^{-1}, \tag{4.5}$$

Q being an arbitrary constant-term matrix with an inverse. From (4.5), $X^2=0$.

From (4.2) we then define $A(t)$ by

$$A(t) = G + tXG = G + tQNQ^{-1}G \tag{4.6}$$

$A^{-1}(t)$ being given by (4.3).

Example. Take

$$N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad Q = \begin{bmatrix} 17 & 6 & 12 \\ 6 & 13 & 24 \\ 12 & 24 & 23 \end{bmatrix},$$

$$X=\begin{bmatrix}24 & 13 & 18\\ 10 & 0 & 14\\ 20 & 0 & 2\end{bmatrix},\qquad G=\begin{bmatrix}25 & 2 & 16\\ 2 & 25 & 10\\ 16 & 10 & 3\end{bmatrix}(=G^{-1}),$$

$$XG=\begin{bmatrix}4 & 7 & 22\\ 6 & 4 & 20\\ 12 & 8 & 14\end{bmatrix},\qquad G^{-1}X=\begin{bmatrix}4 & 13 & 16\\ 4 & 0 & 16\\ 24 & 0 & 18\end{bmatrix},$$

$$A(t)=\begin{bmatrix}25 & 2 & 16\\ 2 & 25 & 10\\ 16 & 10 & 3\end{bmatrix}+t\begin{bmatrix}4 & 7 & 22\\ 6 & 4 & 20\\ 12 & 8 & 14\end{bmatrix},$$

$$A^{-1}(t)=\begin{bmatrix}25 & 2 & 16\\ 2 & 25 & 10\\ 16 & 10 & 3\end{bmatrix}-t\begin{bmatrix}4 & 13 & 16\\ 4 & 0 & 16\\ 24 & 0 & 18\end{bmatrix}.$$

Using these in (4.1) gives

$$C_i=\begin{bmatrix}25+4k_i & 2+7k_i & 16+22k_i\\ 2+6k_i & 25+4k_i & 10+20k_i\\ 16+12k_i & 10+8k_i & 3+14k_i\end{bmatrix}P_i, \tag{4.7}$$

$$P_i=\begin{bmatrix}25+22k_i & 2+13k_i & 16+10k_i\\ 2+22k_i & 25 & 10+10k_i\\ 16+2k_i & 10 & 3+8k_i\end{bmatrix}C_i. \tag{4.8}$$

Take $k_i=p_{i-1,1}+p_{i-1,2}+p_{i-1,3}$, $(k_1=1)$, and encipher CRYPTOGRAPHIC(XX),

$$C_1=\begin{bmatrix}3 & 9 & 12\\ 8 & 3 & 4\\ 2 & 18 & 17\end{bmatrix}\begin{bmatrix}3\\ 18\\ 25\end{bmatrix}=\begin{bmatrix}3\\ 22\\ 1\end{bmatrix}=\begin{bmatrix}\text{C}\\ \text{V}\\ \text{A}\end{bmatrix},\qquad k_2=3+18+25\equiv 20,$$

$$C_2=\begin{bmatrix}1 & 12 & 14\\ 18 & 1 & 20\\ 22 & 14 & 23\end{bmatrix}\begin{bmatrix}16\\ 20\\ 15\end{bmatrix}=\begin{bmatrix}24\\ 10\\ 15\end{bmatrix}=\begin{bmatrix}\text{X}\\ \text{J}\\ \text{O}\end{bmatrix},\qquad k_3=16+20+15\equiv 25,$$

$$C_3=\begin{bmatrix}21 & 21 & 20\\ 22 & 21 & 16\\ 4 & 2 & 15\end{bmatrix}\begin{bmatrix}7\\ 18\\ 1\end{bmatrix}=\begin{bmatrix}25\\ 2\\ 1\end{bmatrix}=\begin{bmatrix}\text{Y}\\ \text{B}\\ \text{A}\end{bmatrix},\qquad k_4=7+18+1\equiv 0,$$

$$C_4=\begin{bmatrix}25 & 2 & 16\\ 2 & 25 & 10\\ 16 & 10 & 3\end{bmatrix}\begin{bmatrix}16\\ 8\\ 9\end{bmatrix}=\begin{bmatrix}14\\ 10\\ 25\end{bmatrix}=\begin{bmatrix}\text{N}\\ \text{J}\\ \text{Y}\end{bmatrix},\qquad k_5=16+8+9\equiv 7,$$

$$C_5 = \begin{bmatrix} 1 & 25 & 14 \\ 18 & 1 & 20 \\ 22 & 14 & 23 \end{bmatrix} \begin{bmatrix} 3 \\ 24 \\ 24 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 18 \end{bmatrix} \quad \begin{bmatrix} \text{C} \\ \text{L} \\ \text{R} \end{bmatrix}.$$

The cipher-text is thus CVA XJO YBA NJY CLR.

To decipher, use (4.8),

$$P_1 = \begin{bmatrix} 21 & 15 & 0 \\ 24 & 25 & 20 \\ 18 & 10 & 11 \end{bmatrix} \begin{bmatrix} 3 \\ 22 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 18 \\ 25 \end{bmatrix} = \begin{bmatrix} \text{C} \\ \text{R} \\ \text{Y} \end{bmatrix}, \textit{ etc.}$$

There is an advantage in using an involutory transformation (4.1), *i.e.*, one such that $A(t) = A^{-1}(t)$. To obtain this we require from (4.3), (4.6),

$$G = G^{-1}, \qquad XG = -GX. \tag{4.9}$$

Define matrix J by

$$\begin{bmatrix} 1 & j_1 & & & & & & & \\ 0 & -1 & & & & & & & \\ & & 1 & j_2 & & & & & \\ & & 0 & -1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 1 & j_q & & \\ & & & & & 0 & -1 & & \\ & & & & & & & a_1 & \\ & & & & & & & & \ddots \\ & & & & & & & & & a_r \end{bmatrix} = J, \tag{4.10}$$

constants $j_1, \cdots, j_q$ arbitrary, and $a_1, \cdots, a_r$ all $= \pm 1$, $(r+2q=n)$.

Then by (4.4), (4.10),

$$J^2 = I, \qquad JN = -NJ. \tag{4.11}$$

Place

$$G = QJQ^{-1}, \qquad X = QNQ^{-1}, \tag{4.12}$$

giving

$$G^2 = I, \qquad XG = -GX, \qquad (X^2 = 0),$$

satisfying (4.9).

From (4.12), $XG = QNJQ^{-1} = -QNQ^{-1} = -X$, since direct calculation shows $NJ = -N$. Hence (4.6) gives

$$A(t) = G - tX = A^{-1}(t), \tag{4.13}$$

which can also be expressed as

$$A(t) = Q(J - tN)Q^{-1} = A^{-1}(t). \tag{4.14}$$

Example. Use N, Q, X of the previous example, and

$$J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

By (4.12), (4.13)

$$G = \begin{bmatrix} 3 & 9 & 10 \\ 6 & 19 & 2 \\ 12 & 14 & 3 \end{bmatrix}, \quad A(t) = \begin{bmatrix} 3+2t & 9+13t & 10+8t \\ 6-10t & 19 & 2+12t \\ 12+6t & 14 & 3+24t \end{bmatrix} = A^{-1}(t).$$

In case $n=2$ it can be verified that

$$A(t) = A^{-1}(t) = \begin{bmatrix} a+bt & c+dt \\ e+ft & -(a+bt) \end{bmatrix}$$

if $a = bcd' \pm 1$, $e = -b^2c(d')^2 \mp 2bd'$, $f = -b^2d'$, and b, c, d are arbitrary (d prime mod 26), and $dd' \equiv 1 \bmod 26$.

The modulus 26 used throughout this paper is not essential. Other moduli can be used with suitable modifications where necessary.

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THE MOTION OF A FLEXIBLE INELASTIC TUBE CONSTRAINED TO MOVE ON A ROUGH CONVEX CURVE

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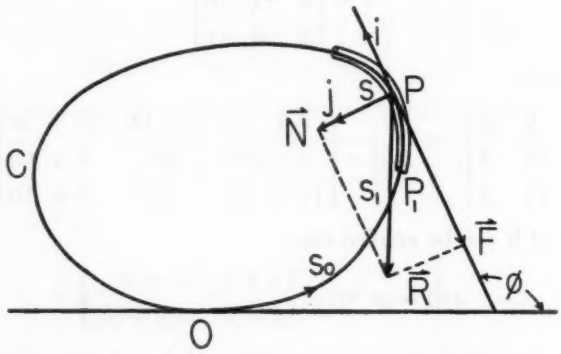
One might well expect that the behavior of a flexible inelastic tube constrained to move on a rough wire would be more complex than that of a bead or particle subjected to the same constraint. Indeed, such is the case. However, it is an agreeable fact that the motion of the tube has some of the same invariant properties possessed by the particle. The case of the particle has already been discussed [4 (pp. 16–20), 2, 3, 5, 6]. Since this topic is a very old one, it is to be expected that some of the results presented here as new to the author may be known to others. For the sake of the reader, a brief and concise derivation of the

known fundamental differential equations (5) is given.

DEFINITION. *The boundary of a bounded plane convex body is called a convex curve (a convex body is a convex set having interior points). A convex curve at each of whose points the curvature exists is called a differentiable convex curve C.*

We restrict ourselves to such curves in this paper. It will be helpful to refer to the following diagram.



1. Formulation of the problem. We assume that C corresponds to a fixed rough wire of length L whose coefficient of friction μ is constant. A flexible inelastic tube* A is constrained to move on C. The tube has constant linear density σ. Furthermore, we assume that *the only force acting on the tube A is the fixed constraint C*; no other external forces including gravity act on A. Let s be the arc length of that portion of C from the fixed point O to an arbitrary point P (see diagram), measuring distances positively on C counterclockwise. Moreover, let s_1 denote the value of s corresponding to the position of the terminal endpoint P_1 of the tube. The angle ϕ is the conventional angle which the tangent to C makes with the positive x-axis through O (see diagram).

Let $\vec{R}$ denote the reaction of the constraint per unit mass of the tube A. This is resolved into normal and tangential components $\vec{N}$ and $\vec{F}$ respectively. The law of friction is $|\vec{F}| = \mu |\vec{N}|$. The scalar tension in the tube is denoted by T. We define $\vec{T} \equiv T\vec{i}$, where $\vec{i}$ is the unit tangent vector to C at P (see diagram). The tension and the reaction are functions of s and s_1. We assume that $\vec{R}$ is a continuous function of s for $s_1 \leqq s \leqq s_1 + a$, $s_1 \geqq s_0$, where a is the length of the tube A, and where $a < L$.

We adopt the following initial conditions

* It will assist the reader to regard the tube as a hollow slender flexible string which is strung on the wire C. We idealize the situation so that the tube is one-dimensional, inelastic and flexible.

(1) $$t = 0, \qquad s_1 = s_0, \qquad \phi(s_0) = \phi_0, \qquad v \equiv ds_1/dt = v_0 > 0.$$

Let P be a fixed point in the tube A, and let b denote the length of the subarc of A which has the points P and P_1 as its endpoints. Designate the speed of P by $v(s_1, b)$, and denote the corresponding vector velocity of P by $\vec{V}(s_1, b)$. The speed of the terminal endpoint P_1 of the tube A is denoted by $v(s_1)$, so that $v(s_1) = v(s_1, 0)$.

2. Conclusions. Since the tube is inextensible, we have $v(s_1, b) = v(s_1) = ds_1/dt$ for $0 \leqq b \leqq a$, $s_1 \geqq s_0$.

THEOREM 1. *The speed $v(s_1)$ of the tube A subject to the initial conditions* (1) *has the form*

(2) $$v(s_1) = v_0\left(\int_{s_0}^{s_0+a} e^{\mu\phi(s)}ds\right) \Big/ \left(\int_{s_1}^{s_1+a} e^{\mu\phi(s)}ds\right),$$

where a is the length of the tube A, and where s_1 is the arc length of that portion of C from the fixed point O to the terminal endpoint P_1 of A.

Proof. Select any *fixed* point P in the tube A at the time t, and let ΔA be a segment of A of length $|\Delta s|$ and having P as one of its endpoints. The fundamental axioms of Newtonian mechanics when applied to the tube include the following statement.

Axiom. There exists a vector function $\vec{\epsilon}(s, \Delta s)$, $s_1 \leqq s \leqq s_1 + a$ such that $\vec{\epsilon} \to 0$ as $\Delta s \to 0$, and such that

(3) $$\sigma \Delta s \frac{d\vec{V}(s_1, b)}{ds_1}\frac{ds_1}{dt} = \int_s^{s+\Delta s} \sigma \vec{R} ds + \Delta \vec{T} + \vec{\epsilon}\Delta s, \qquad s = s_1 + b,$$

where $\Delta \vec{T}$ is the difference of the tensions acting at the endpoints of ΔA (see section 1 for notations). Upon dividing the first of equations (3) by Δs, we observe that $\partial \vec{T}/\partial s$ exists since $\vec{R}$ is a continuous function of s for $s_1 \leqq s \leqq s_1 + a$. Hence,

(4) $$\sigma \frac{d\vec{V}(s_1, b)}{dt} = \sigma \vec{R}(s_1 + b, s_1) + \frac{\partial \vec{T}(s, s_1)}{\partial s}\Big|_{s=s_1+b}$$

holds for $s_1 \leqq s \leqq s_1 + a$, $s_1 \geqq s_0$, $0 \leqq b \leqq a$. Since $\vec{T} \equiv T\vec{i}$, we must have

$$\frac{\partial \vec{T}}{\partial s} = \frac{\sigma T}{\partial s}\vec{i} + \frac{T}{\rho}\vec{j}, \qquad 1/\rho = d\phi/ds,$$

where $\vec{i}$ and $\vec{j}$ are unit vectors along the tangent and normal respectively to C

at P. Let $\vec{F}=-F\vec{i}$, $\vec{N}=N\vec{j}$, so that $|\vec{F}|=F$, $|\vec{N}|=N$. Since $\vec{V}(s_1, b)=v(s_1)\vec{i}(s_1+b)$, upon resolving equation (4) into tangential and normal components, we obtain

$$\sigma \frac{dv(s_1)}{ds_1} v(s_1) = -\sigma F(s, s_1) + \frac{\partial T(s, s_1)}{\partial s}, \tag{5}$$

$$\frac{\sigma v^2(s_1)}{\rho(s)} = \sigma N(s, s_1) + \frac{T(s, s_1)}{\rho(s)},$$

where $1/\rho=d\phi/ds$. Since $F=\mu N$, equations (5) imply that

$$\frac{\partial T(s, s_1)}{\partial s} + \mu \frac{T(s, s_1)}{\rho(s)} = \sigma\left[v(s_1) \frac{dv(s_1)}{ds_1} + \mu \frac{v^2(s_1)}{\rho(s)}\right]. \tag{6}$$

Moreover, since the ends of the tube A are free, condition (3) implies that

$$T(s_1, s_1) = 0, \qquad T(s_1 + a, s_1) = 0. \tag{7}$$

Equation (6) and the first of (7) imply that

$$T(s, s_1) = e^{-\mu(\phi-\phi_1)} \int_{s_1}^{s} \left(\sigma v \frac{dv}{ds_1} + \mu \frac{\sigma v^2}{\rho}\right) e^{\mu(\phi-\phi_1)} ds, \tag{8}$$

where $\phi=\phi(s)$, $\phi_1=\phi(s_1)$, $v=v(s_1)$ and $1/\rho=d\phi/ds$. The integral in (8) exists since $1/\rho$ is integrable [1]. The condition $T(s_1+a, s_1)=0$ can then be satisfied if and only if the integral in (8) is zero for $s=s_1+a$. Since $v=v(s_1)$, and since $d\phi/ds=1/\rho$, the vanishing of the integral in (8) for $s=s_1+a$ implies

$$v \frac{dv}{ds_1} = \frac{-v^2\left[e^{\mu\phi(s_1+a)} - e^{\mu\phi(s_1)}\right]}{\int_{s_1}^{s_1+a} e^{\mu\phi} ds}. \tag{9}$$

Since $v_0>0$, if $v\neq 0$ for $t\geqq 0$, the unique solution of (9) subject to the initial conditions (1) satisfies the equation

$$\frac{dv}{ds_1} = -v \frac{J'(s_1)}{J(s_1)}, \quad \text{where} \quad J(s_1) \equiv \int_{s_1}^{s_1+a} e^{\mu\phi} ds, \qquad v \neq 0. \tag{10}$$

The solution of (10) subject to (1) is precisely $v=v_0 J(s_0)/J(s_1)$, which is the same as that given in equation (2) of the theorem. Equation (2) implies that $v\to 0$ and $s\to\infty$ as $t\to\infty$. Hence, our assumption that $v\neq 0$ for $t\geqq 0$ is a perfectly valid one, since (9) has a unique solution satisfying (1).

THEOREM 2. *The speed with which the tube returns to its initial position is independent of the convex curve C and of the length of the tube A (no gravity acts on A; see formulation of the problem).*

Proof. The desired speed is $v(s_0+L)$, where L is the length of C. Since

$$J(s_0 + L) \equiv \int_{s_0+L}^{s_0+L+a} e^{\mu\phi(s)}ds = \int_{s_0}^{s_0+a} e^{\mu\phi(L+x)}dx,$$

and since $\phi(L+x)=\phi(x)+2\pi$, we get $J(s_0+L)=e^{2\pi\mu}J(s_0)$, so that

$$v(s_0 + L) = v_0 \frac{J(s_0)}{J(s_0 + L)} = \frac{v_0 J(s_0)}{e^{2\pi\mu}J(s_0)} = v_0 e^{-2\pi\mu}.$$

In the following theorem we inquire as to what happens to the speed v of the tube as a function of s_1 and the length a. Hence, the left member of (2) is written in the form $v(s_1, b, a)$ so that we have $v(s_1, b, a)=v(s_1, 0, a)$.

THEOREM 3. *The following limit*

$$v \equiv \lim_{a\to 0} v(s_1, 0, a) = v_0 e^{\mu(\phi_0-\phi_1)} \tag{11}$$

holds, where $\phi_1=\phi(s_1)$, $\phi_0=\phi(s_0)$, *and where* a *is the length of the tube. This value is the same as that of a particle constrained to* C *and satisfying* (1).

Proof. The limit in (11) follows immediately by an application of an appropriate form of L'Hospital's rule to the indeterminate form (2) for $a=0$. The second sentence in the theorem follows immediately from the theory for a particle ([1], p. 17).

THEOREM 4. *If* C *is a circle of radius* r, *then the speed* $v(s_1)$ *of the tube is the same as that of a particle, so that* $v(s_1)$ *has the form given in* (11).

Proof. Equation (2) can be written in the form

$$v = v_0 e^{\mu(\phi_0-\phi_1)}\left(\int_{s_0}^{s_0+a} e^{\mu(\phi-\phi_0)}ds\right)\Big/\left(\int_{s_1}^{s_1+a} e^{\mu(\phi-\phi_1)}ds\right). \tag{12}$$

Since for a circle we have $\phi-\phi_1=(s-s_1)/r$, $\phi-\phi_0=(s-s_0)/r$, the two integrals in (12) are equal. This implies that $v(s_1)$ has the form given in (11).

THEOREM 5. *The tension* $T(s, s_1)$ *in a tube of positive length* a *is identically zero for* $s_1\leqq s\leqq s_1+a$, $s_0\leqq s_1\leqq s_0+L$ *if and only if* C *is a circle.*

Proof. Suppose $T(s, s_1)\equiv 0$ on the intervals indicated. Since $v(s_1, b)=v(s_1)$ for $0\leqq b\leqq a$, equation (6) and the assumption $T(s, s_1)\equiv 0$ imply that

$$\frac{dv(s_1, b)}{ds_1} + \mu\frac{v(s_1, b)}{\rho(s_1 + b)} = 0, \qquad v(s_0, b) = v_0. \tag{13}$$

The solution of (13) is

$$v(s_1, b) = v_0 e^{\mu[\phi(s_0-b)+\phi(s_1-b)]}. \tag{14}$$

Since $v(s_1, b) = v(s_1, 0)$ for $0 \leqq b \leqq a$, equation (14) implies

$$\phi(s_0 + b) - \phi(s_1 + b) = \phi(s_0) - \phi(s_1).$$

This implies that

$$\frac{\phi(s_1 + b) - \phi(s_1)}{b} = \frac{\phi(s_0 + b) - \phi(s_0)}{b} \qquad b \neq 0,$$

so that

$$\left.\frac{d\phi}{ds}\right|_{s=s_1} = \left.\frac{d\phi}{ds}\right|_{s=s_0}, \qquad s_1 \geqq s_0.$$

Hence, the curvature $d\phi/ds$ is a constant, and since C is a plane convex curve, it must be a circle.

Conversely, if C is a circle, it is a simple matter to verify that the integrand in (8) must vanish, so that $T(s, s_1) \equiv 0$ on the intervals indicated.

3. Some thought provoking questions. The following questions have puzzled the writer for some time; they do not appear to possess simple answers.

(a) Suppose the tube has a finite breaking strength. What is the range of values of the initial speed v_0 so that the tube will remain unbroken during its subsequent motion?

(b) What can one say about the number of places where the tube can break apart simultaneously? What happens to the different parts of a tube which breaks apart during its motion?

(c) How does the period of time required for the tube to complete one journey vary with the length of the tube?

(d) What can be said about the speeds of tubes of different lengths other than that which was said in Theorem 2?

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THE PASCAL LINE AND ITS GENERALIZATIONS

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1. Introduction. It is now about 300 years since Pascal (1623–1662) first enunciated his famous theorem concerning hexagons inscribed in conic sections and an extensive literature has since grown up around it. Many references to this literature will be found in [2], [5], and [7]. It would seem hardly possible to add much that is new to the subject at this stage, but the converse of Pascal's theorem seems to have attracted very little attention and a study of the theorem from this point of view leads to many interesting results. Our main aim in this article is to see how far Pascal's theorem can be extended to higher plane curves or to three dimensional surfaces and the converse of the theorem proves to be very useful in this connection.

Clifford (1845–1879) studied the intersections of coplanar lines in one of his earliest papers [3] and obtained a result related to the one we give here in Section 3. He does not seem to have gone beyond this case although at the end of the first section of this paper he writes "The present communication is confined mainly to the case $n=4$; in a second communication we hope to notice some peculiarities of the higher plane case and to state the *true* analogues of Pascal's theorem in the Geometry of Higher Dimensions." No doubt Clifford's untimely death and the long illness which preceded it prevented him from returning to this topic.

Pascal's theorem has been extended to three dimensions in at least two ways. One extension is stated by Chasles in his monumental work on geometry [2] and is proved by Salmon ([8] Sec. 144, p. 141.) This was rediscovered by Court [4] and recently extended to n dimensions by Bottema [1]. Another extension, whose converse we shall discuss in Section 6, appears to have been first stated and proved by Salmon ([8], Sec. 144b, p. 143.)

I am indebted to the referee for many of the references above.

2. The converse of Pascal's theorem. We first prove the following result:

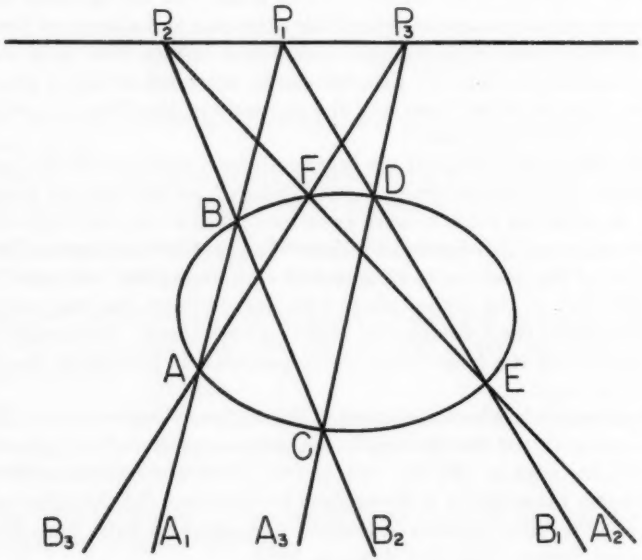
Given a conic, equation $C_2=0$, and a coplanar line, $L=0$, it is always possible to inscribe a hexagon in the conic whose Pascal line is $L=0$.

Let $A_m=a_mx+b_my+1$, $B_m=c_mx+d_my+1$, and let the point of intersection of the lines $A_m=0$, $B_n=0$ be denoted by (A_m, B_n). In order to prove the theorem we must establish the following identity:

$$C_2L = aA_1A_2A_3 + bB_1B_2B_3, \tag{1}$$

where a and b are constants. For each line $A_m=0$ and $B_n=0$ we have two constants at our disposal and so with a and b we have altogether fourteen constants at our disposal. Against this (1) is a cubic equation and a cubic in x and y possesses ten terms, so that ten equations must be satisfied among the coefficients of (1) if it is to be an identity. Hence we can establish (1) with four constants to spare.

Again, the nine points of intersection of the three A lines with the three B lines in (1) must also lie either on the conic or on $L=0$. Consider the three intersections (A_1, B_1), (A_1, B_2), and (A_1, B_3). Since the line $A_1=0$ can cut the conic in only two points, two of these, (A_1, B_2) and (A_1, B_3) will lie on the conic and the other on the line $L=0$. Similarly (A_2, B_1) and (A_2, B_3) will lie on the conic and (A_2, B_2) will lie on the line and finally (A_3, B_1) and (A_3, B_2) will lie on the conic and (A_3, B_3) on the line.



This completes the proof that a hexagon can be inscribed in a given conic so as to have a given line as its Pascal line, and we note that this can be done with four arbitrarily chosen constants.

To illustrate the theorem let us fix the line $A_1=0$; using up two of our four disposable constants; and also the directions of the lines $B_2=0$ and $B_3=0$; using up one constant for each direction. In Figure 1 the line $A_1=0$ meets the prescribed conic at A and B and the prescribed Pascal line at P_1. Suppose that the $B_2=0$ line, which must pass either through A or through B, passes through B. Since its direction is known we can find its other intersection with the conic at C, say, and also its intersection with the prescribed Pascal line, P_2, say. The line $B_3=0$ must now pass through A and, its direction being known, will cut the conic again at a point, F, say, and the Pascal line at P_3. Evidently F must be (A_2, B_3) and P_2 must be (A_2, B_2), so that P_2F is the line $A_2=0$. Let the join of P_2 and F meet the conic again at E. Similarly the join of P_3 and C must be the line $A_3=0$ and if this cuts the conic again at D then we have constructed a

hexagon $ABCDEF$, inscribed in the given conic and having $L=0$ as its Pascal line, as required. DEP_1 is the line $B_1=0$.

3. Extension to cubic curves. We now prove the following extension of this converse of Pascal's theorem:

Given a cubic curve, equation $C_3=0$, and any coplanar line, $L=0$, we can inscribe an octagon in the curve having the following properties. Using the terminology of Section 2, if the octagon consists of the lines $A_m=0$, $m=1, 2, 3, 4$, and $B_n=0$, $n=1, 2, 3, 4$, then the twelve points of intersection (A_m, B_n), $m\neq n$, lie on the cubic and the four points of intersection (A_m, B_m), $m=1, 2, 3, 4$, lie on the line $L=0$.

To prove this we must establish the identity

$$C_3L = aA_1A_2A_3A_4 + bB_1B_2B_3B_4, \tag{2}$$

where $C_3=0$ is the equation of the cubic curve, $L=0$ that of the line, and a and b are constants. A cubic in x and y has fifteen terms and against this we have eighteen constants (two for each line, a and b) at our choice. Hence (2) can be established with three constants still at our disposal.

All the sixteen intersections of the lines $A_m=0$, $m=1, 2, 3, 4$, with the lines $B_n=0$, $n=1, 2, 3, 4$, must evidently lie on either the cubic curve or on $L=0$. Since $A_1=0$ cuts the cubic in three points only we must have (A_1, B_1) on the line $L=0$ and (A_1, B_2), (A_1, B_3), (A_1, B_4) on the cubic. On dealing similarly with the other three A lines we see that of the sixteen points of intersection twelve are on the cubic and four are on the line $L=0$, as outlined in the enunciation of the theorem.

The figure is difficult to draw since we do not have enough degrees of freedom. We have three available constants and so we may choose the line $A_1=0$, using up two constants, and the direction of $B_2=0$ (which must pass through one of the three intersections of $A_1=0$ with the cubic) using up the third constant. From this stage we cannot proceed any further geometrically; we must actually use the ten coefficient equations required to establish (2) in order to compute the unknown constants of the A and B lines which still remain to be drawn.

4. Extension to quartic curves. For the plane quartic curve we can prove the following theorem:

Given a plane quartic curve whose equation is $C_4=0$ and any coplanar line, equation $L=0$, we can inscribe in the quartic a ten-sided polygon whose sides can be grouped into two sets of five lines, $A_m=0$, $m=1, 2, 3, 4, 5$, and $B_n=0$, $n=1, 2, 3, 4, 5$, with the following properties. The twenty intersections (A_m, B_n), $m\neq n$, lie on the quartic curve and the five intersections (A_m, B_m), $m=1, 2, 3, 4, 5$, lie on the line $L=0$.

The proof, as before, consists in the establishment of the identity

$$C_4L = aA_1A_2A_3A_4A_5 + bB_1B_2B_3B_4B_5, \tag{3}$$

where a and b are constants. A quintic in x and y has twenty-one terms of varying degrees and on the right hand side of (3) we have twenty-two constants at our disposal (two for each line and a, b). Hence (3) can be established with one free constant still at our disposal. Evidently the five A_m lines intersect the five B_n lines in twenty-five points all of which lie either on the quartic or on $L=0$. Among the intersections of $A_1=0$ with the five B_n lines, four will lie on the quartic and one, (A_1, B_1), say, will lie on $L=0$. On dealing similarly with the other A_m lines we see that the twenty-five intersections of the A_m lines with the B_n lines lie twenty on the quartic and five on the line $L=0$ in the manner described above.

Once again the figure is difficult to draw since we have only one free constant. Any line in the plane of the quartic can act as a "generalized Pascal line" for some ten-sided polygon inscribed in the quartic. But, once one of the vertices of the polygon is chosen, then the remaining vertices are determined. There may be several such polygons once one vertex is chosen but the figure is too difficult to draw.

5. The impossibility of extension beyond the quartic. A generalization of the previous results would be as follows:

Given a quintic, $C_5=0$, and a coplanar line, $L=0$, then a twelve-sided polygon, $A_m=0$, $m=1, 2, 3, 4, 5, 6$, $B_n=0$, $n=1, 2, 3, 4, 5, 6$, can be inscribed in the quintic so that thirty of the intersections of the A_m with the B_n lines lie on the quintic and six intersections, (A_m, B_m) $m=1, 2, 3, 4, 5, 6$, lie on the line $L=0$.

In order to prove this we must attempt to establish the identity

$$C_5L = aA_1A_2A_3A_4A_5A_6 + bB_1B_2B_3B_4B_5B_6, \tag{4}$$

where a and b are constants. Since a sextic in x and y has twenty-eight terms (4) can hold only if twenty-eight equations among the coefficients are satisfied. But we have only twenty-six disposable constants, two for each of the A_m and B_n lines and also a and b, so that (4) cannot be established in general. In other words, no line in the plane of a quintic can act as a "generalized Pascal line" for a polygon inscribed in it. Thus the Pascal theorem cannot be extended to quintics and a similar argument shows that it cannot be extended to any curve of degree greater than four.

This means that the type of generalization considered here cannot be carried beyond the quartic. Of course other types of generalization are possible, of which one example is given in [6].

6. Pascal's theorem in three dimensions. We now consider the extension of Pascal's theorem to ruled quadric surfaces, by which we mean hyperboloids of one sheet and hyperbolic paraboloids. Degenerate cases, such as cones, cylinders and pairs of planes are excluded.

Let $P_m=a_mx+b_my+c_mz+1$, $Q_n=d_nx+e_ny+f_nz+1$ and let us denote the line of intersection of the planes $P_m=0$ and $Q_n=0$ by (P_m, Q_n). We shall consider

hexagons inscribed in ruled quadric surfaces, but the hexagons will be constructed of six planes, instead of six lines, and they will intersect the surfaces in generators. To show that such a construction is possible in practice let H denote the ruled quadric and let X_1, X_2, X_3 be any three points on H, chosen so that no two lie on the same generator. The tangent planes at these points, equations $P_1=0$, $P_2=0$, and $P_3=0$, respectively, say, can form three of the planes of the hexagon inscribed in H. Another three planes can be found as follows. The plane $P_1=0$ touches H at X_1 and intersects H in two generators, g_1 and g_1', say. Similarly the planes $P_2=0$ and $P_3=0$ intersect H in the generators g_2 and g_2', g_3, and g_3' respectively, where g_1, g_2, g_3 are all generators of one kind and g_1', g_2', g_3' are all generators of the other kind. Let g_1 and g_2' intersect at Y_3, g_2 and g_3' at Y_1 and g_3 and g_1' at Y_2. Then the tangent planes at Y_1, Y_2 and Y_3 can form the other three planes of the hexagon inscribed in the ruled surface H. If the tangent planes at Y_1, Y_2, Y_3 have equations $Q_1=0$, $Q_2=0$, $Q_3=0$, respectively, then evidently g_1' is the line of intersection (P_1, Q_2), g_1 is the line (P_1, Q_3), and there are corresponding results for the other four generators. Thus among the nine lines of intersection of the three P planes with the three Q planes six are generators of H.

The following generalization of Pascal's theorem to three dimensions is proved by Salmon ([8], Sec. 144b, p. 143).

THEOREM. *If among the nine lines of intersection of the three planes $P_m=0$, $m=1, 2, 3$, with the three planes $Q_n=0$, $n=1, 2, 3$, six lines, (P_m, Q_n), $m\neq n$, lie on a ruled quadric surface then the remaining three lines, (P_m, Q_m), $m=1, 2, 3$, must be coplanar.*

We now consider the converse, which is as follows:

Given a ruled quadric, equation $H=0$, and a plane, equation $P=0$, we can always construct a hexagon of six planes, $P_m=0$, $m=1, 2, 3$ and $Q_n=0$, $n=1, 2, 3$, so that six of the nine lines of intersection of the P planes with the Q planes, (P_m, Q_n), $m\neq n$, are generators of $H=0$ and the remaining three, (P_m, Q_m), $m=1, 2, 3$, lie on the plane $P=0$.

To prove this we must establish the identity

$$HP = pP_1P_2P_3 + qQ_1Q_2Q_3, \tag{5}$$

where p and q are constants and H and P are given. Now a cubic in x, y, z possesses twenty terms of varying degrees so that (5) can hold only if twenty coefficient equations are satisfied. Against this we have twenty disposable constants on the right hand side, three for each plane and p, q, so that (5) can be established, but without constants to spare.

Evidently each of the nine lines of intersection of the three P planes with the three Q planes must lie either on $H=0$ or on $P=0$. Consider the three lines (P_1, Q_1), (P_1, Q_2) and (P_1, Q_3). Since $P_1=0$ cuts $H=0$ in a conic section two of these lines will lie on $H=0$ and the third, (P_1, Q_1), say, will lie on $P=0$. By dis-

cussing the other six lines of intersection similarly we see that six of the nine intersections lie on $H=0$ and three on $P=0$, in the manner described in the enunciation of the theorem.

Since there are no free disposable constants it follows that given the plane $P=0$ we can inscribe in $H=0$ only a finite number of hexagons whose plane sides have three lines of intersection on $P=0$. This is in complete contrast with the two-dimensional theorem of Section 2 in which for any given Pascal line an infinite number of hexagons can be inscribed in a given conic.

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MATHEMATICAL NOTES

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A REMARK ON FINITELY ADDITIVE MEASURES

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An *algebra* of sets is a family of subsets of a set S which contains the void set ϕ and is closed under complementation and finite unions. By a *finitely additive measure* we shall mean a real-valued function μ defined on an algebra Σ of sets such that

$$\mu(E_1 \cup \cdots \cup E_n) = \sum_{i=1}^{n} \mu(E_i)$$

for every finite family $E_1, \cdots, E_n$ of mutually disjoint sets in Σ. We say μ is *bounded* if there exists a real number M such that $|\mu(E)| \leqq M$, $E \in \Sigma$.

If μ is nonnegative or merely bounded above or below, then it is bounded,

[1]. If Σ is a σ-algebra and μ is countably additive on Σ, then it is well known that μ must be bounded, [2, p. 17]. In a recent paper [1], E. Hewitt has remarked that it is not known whether a finitely additive measure whose domain is a σ-algebra is necessarily bounded. We shall answer this question negatively by the following theorem.

THEOREM. *If Σ is any infinite algebra of sets, there exists a finitely additive measure defined on Σ which is unbounded.*

Proof. We observe first that there exists in Σ an infinite sequence $\{E_n\}$ of mutually-disjoint nonvoid sets. For let $\{F_n\}$ be any sequence of distinct nonvoid sets in Σ. Let $E \in \Sigma$, $E \neq \phi$, $E \neq S$. We assert that one of the two sequences $\{E \cap F_n\}$ or $\{E' \cap F_n\}$ contains infinitely many distinct sets. For suppose there exists an integer N such that the sets $E \cap F_n$ all coincide for $n \geqq N$. Then since $F_n = (E \cap F_n) \cup (E' \cap F_n)$, all of the sets $E' \cap F_n$ must be distinct for $n \geqq N$. Thus there exists a nonvoid set E_1 in Σ such that infinitely many of the sets $E_1' \cap F_n$ are distinct. Applying the same argument with S replaced by E_1', we construct a nonvoid set $E_2 \subseteq E_1'$, such that infinitely many of the sets $(E_1' - E_2) \cap F_n$ are distinct. The construction proceeds by induction.

Now select points $t_n \in E_n$, and for each n define the measure ρ_n by the formulas $\rho_n(E) = 1$ if $t_n \in E$, $\rho_n(E) = 0$ if $t_n \notin E$. Let

$$\nu = \sum_{n=1}^{\infty} \frac{\pi^n}{4^n} \rho_n.$$

Then ν is finitely additive (in fact the natural extension of ν to the σ-algebra generated by Σ is countably additive). The sequence $\mathfrak{A} = \{4^{-n}\pi^n\}$ is linearly independent over the rational field, and we may imbed $\mathfrak{A}$ in a Hamel base $\mathfrak{B}$ for the real numbers, [2, p. 100]. Now define the real valued function α on $\mathfrak{B}$ in the following way: on $\mathfrak{A}$ define $\alpha(4^{-n}\pi^n) = n$, and define $\alpha(b) = 0$ for $b \in \mathfrak{B} - \mathfrak{A}$. We extend α to all real numbers by linearity. For if x is any real number, there exist unique elements $b_1, \cdots, b_m$ in $\mathfrak{B}$ and rationals $r_1, \cdots, r_m$ so that $x = \sum_{i=1}^{m} r_i b_i$. The formula $\alpha(x) = \sum_{i=1}^{m} r_i \alpha(b_i)$ gives the required extension. The set function μ defined by

$$\mu(E) = \alpha(\nu(E)), \qquad E \in \Sigma,$$

is finitely additive. Since

$$\mu(E_n) = \alpha(\nu(E_n)) = \alpha(4^{-n}\pi^n) = n, \qquad n = 1, 2, \cdots,$$

we see μ is unbounded.

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ON THE PRODUCT OF TWO CONTINUOUS FUNCTIONS OF BOUNDED VARIATION

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Introduction and preliminaries.

DEFINITION 1. *A function T on an interval I will be termed a CBV function if and only if T is both continuous and of bounded variation on I.*

We attempt in this paper to give a necessary and sufficient condition for a CBV of a CBV to be a CBV. $T_1(x)=y$ will denote a continuous function on $[\alpha, \beta]=I_x$ with images satisfying $a\leqq y\leqq b$. I_y denotes $[a, b]$. $T_2(x)=z$ will denote a second continuous transformation on $I_y=[a, b]$ with images satisfying $c\leqq z\leqq d$. $[c, d]=I_z$. $T_{21}(x)=T_2\{T_1(x)\}$ for $x\in I_x$. Hence $T_{21}(x)$ will always be continuous from I_x to I_z. $m(S)$ will denote the Lebesgue measure of a measurable set S.

DEFINITION 2. *The multiplicity function $N(y, T_1, I_x)$ is defined as follows:*

$$N(y, T_1, I_x) \equiv \begin{cases} n \text{ if } T_1^{-1}(y) \text{ consists of } n \text{ points for } n = 0, 1, \cdots, \\ \infty \text{ if } T_1^{-1}(y) \text{ is infinite.} \end{cases}$$

DEFINITION 3. *If $S_x\subset I_x$, then*

$$N(y, T_1, S_x) = \begin{cases} n \text{ if } T_1^{-1}(y)\cap S_x \text{ has } n \text{ points for } n = 0, 1, \cdots, \\ \infty \text{ if } T_1^{-1}(y)\cap S_x \text{ is infinite.} \end{cases}$$

$N(z, T_2, I_y)$ and $N(z, T_{21}, I_x)$ are similarly defined.

THEOREM 1. *If $y=T(x)$ is continuous on I_x, then $N(y, T, I_x)$ is a Borel measurable function of y (R, 36).**

THEOREM 2. *If $y=T(x)$ is continuous on I_x, then $T(x)$ is BV if and only if $N(y, T, I_x)$ is summable on the y-axis (R, 196).*

Main Results.

LEMMA 1. *Let T_1 be CBV on I_x, $T_2(y)$ be CBV on I_y and $T_1(I_x)\subset I_y$. Let $T_{21}(x)$ be CBV on I_x. Let $\overline{Y}$ be $\{y\,|\,N(y, T_1, I_x)=\infty\}$. Then $m\{T_2(\overline{Y})\}=0$.*

Proof. Let $y_1\in I_y$ and $z_1=T_2(y_1)$. $T_1^{-1}(y_1)$ is the set of all points x such that $T_1(x)=y_1$. $T_{21}^{-1}(z_1)$ is the set of all x such that $T_{21}(x)=z_1$. Then $T_1^{-1}(y_1)\subset T_{21}^{-1}(z_1)$. Hence $N(y_1, T_1, I_x)\leqq N(z_1, T_{21}, I_x)$. But for y in $\overline{Y}$, $N(y, T_1, I_x)=\infty$. Thus if $z\in T_2(\overline{Y})$, then $N(z, T_{21}, I_x)=\infty$. Then if $m\{T_2(\overline{Y})\}>0$, it follows that $\int_{I_z} N(z, T_{21}, I_x)=\infty$. Then T_{21} is not BV on I_x by Theorem 2, a contradiction.

Again, let $T_1(x)$ be CBV on I_x and $T_2(y)$ be CBV on I_y and $T_1(I_x)\subset I_y$. Let T_{21} be the product function on I_x. Let $Y_i\equiv\{y\,|\,N(y, T_1, I_x)=i\}$ $i=0, 1, \cdots$. Let $Z_{ji}\equiv\{z\,|\,T_2^{-1}(z)\cap Y_i \text{ has } j \text{ points}\}$ $j=0, 1, \cdots$. The sets Z_{ji} are not neces-

* (R, 36) refers to Rado [3].

sarily disjoint. It is clear that $Z_{ji} \equiv \{z \mid N(z, T_2, Y_i) = j\}$. Now Y_i is a Borel set since $N(y, T_1, I_x)$ is a Borel measurable function by Theorem 1. It follows that $T_2(Y_i)$ is Lebesgue measurable (R, 31). $N(z, T_2, Y_i)$ is a measurable function (R, 201) and hence Z_{ji} is a measurable set.

THEOREM 3. *Let T_1 be CBV on I_x, T_2 be CBV on I_y, $T_1(I_x) \subset I_y$. Then T_{21} is CBV on I_x if and only if both of the following conditions hold:*

(a) $m\{T_2(\overline{Y})\} = 0$, (b) $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} ijm(Z_{ij}) < \infty$.

Proof. Sufficiency. Assume that (a) and (b) both hold. Now $z \in Z_{ji}$ implies $N(z, T_2, Y_i) = j$. Since T_2 is CBV, $N(z, T_2, I_y)$ is finite a.e. and hence $N(z, T_2, Y_i)$ is finite a.e. Therefore for each i,

$$\sum_{j=0}^{\infty} jm(Z_{ji}) = \int_{I_x} N(z, T_2, Y_i)dz$$

and

$$\sum_{i=0}^{\infty} i\left\{\sum_{j=0}^{\infty} jm(Z_{ji})\right\} = \sum_{i=0}^{\infty} i \int_{I_x} N(z, T_2, Y_i)dz.$$

By (b) we have $\sum_{i=0}^{\infty} i \int_{I_x} N(z, T_2, Y_i)dz < \infty$. It is easy to see that $N(z, T_{21}, I_x) = \sum_{i=1}^{\infty} iN(z, T_2, Y_i)$ for all z not in $\overline{Z} \cup T_2(\overline{Y})$ where $\overline{Z} \equiv \{z \mid N(z, T_2, I_y) = \infty\}$. But $m(\overline{Z}) = 0$ and $m\{T_2(\overline{Y})\} = 0$. Hence

$$\int_{I_x} N(z, T_{21}, I_x)dz = \int_{I_x} \sum_{i=0}^{\infty} iN(z, T_2, Y_i)dz = \sum_{i=0}^{\infty} i \int_{I_x} N(z, T_2, Y_i)dz < \infty.$$

This yields $T_{21}(x)$ BV on I by Theorem 2.

Necessity. Assume that $T_{21}(x)$ is CBV on I_x. (a) follows from Lemma 1. From Theorem 2

$$\infty > \int_{I_x} N(z, T_{21}, I_x)dz = \int_{I_x} \sum_{i=0}^{\infty} iN(z, T_2, Y_i)dz = \sum_{i=0}^{\infty} i \int_{I_x} N(z, T_2, Y_i)dz$$

$$= \sum_{i=0}^{\infty} i\left\{\sum_{j=0}^{\infty} jm(Z_{ji})\right\}.$$

Hence (b) follows.

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THE NELSON SLIDE RULE

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The device to be described here, known locally as the Nelson slide rule, is the by-product of an engineering student's study of mechanical computing methods. This device is presented not as a slide rule competitive with the conventional type but as an interesting study in numerical relationships. It was developed by Alfred M. Nelson while a senior in mechanical engineering at the University of Washington a few years ago. Mr. Nelson is now professionally occupied in the development of computing equipment and it has devolved upon one of his former professors to bring his slide rule to the printed page.

The conventional slide rule multiplies two numbers by adding their logarithms. Toward this end the sliding and stationary scales are each graduated logarithmically, *i.e.*, so that the distance from the left end of the scale to any given number is proportional to the logarithm of that number.

The Nelson slide rule multiplies quite differently and its scales are not graduated logarithmically. Rather, simple nonlinear scales are provided for the numbers to be multiplied and the product is read on a uniformly graduated scale. The computation is based upon relationships among "triangular" integers, where a triangular integer is the sum of all the integers from one through any selected integer. In the following discussion, *triangular a* means the sum of the integers from one through a.

This slide rule multiples two numbers a and b by subtracting triangular $a-b$ from triangular a, and then adding triangular $b-1$. Justification of this procedure will follow discussion of the slide rule and its operation.

The Nelson slide rule is shown in Figure 1, set for multiplying 8 times 3. It is seen to include two stationary scales and two additional scales on a sliding member.

The upper stationary scale, from which the desired product ab is read, is uniformly graduated.

The lower stationary scale, for the multiplier a, is graduated so that the distance from the left end of the scale to any integer a is proportional to triangular a. Accordingly, starting from zero at the left end of the scale, the lengths of successive increments are in the proportion of 1, 2, 3, 4, *etc.* The increment from zero to 1 is the same length as the increment between any two successive integers on the upper stationary scale.

The lower sliding scale, for the term $a-b$, is graduated identically to the lower stationary scale.

The upper sliding scale, for the multiplicand b, is graduated so that the distance from the left end of the scale to any integer b is proportional to triangular $b-1$. The graduation of this scale is identical to the two lower scales, but the numbers on the corresponding graduations are greater by one than on the lower scales.

To multiply with the Nelson slide rule, the multiplier a (the larger of the two

numbers being multiplied) is selected on the lower stationary scale. The difference $a-b$ is selected on the lower sliding scale and is placed opposite the selected a. The multiplicand b is selected on the upper sliding scale, and the product ab is read directly opposite on the upper stationary scale. Thus, in Figure 1, the multiplier is selected to be 8 on the lower stationary scale, the multiplicand is selected to be 3 on the upper sliding scale, the difference of 5 is set on the lower sliding scale opposite 8 on the lower stationary scale, and the product 24 is read on the upper stationary scale opposite the multiplicand 3.

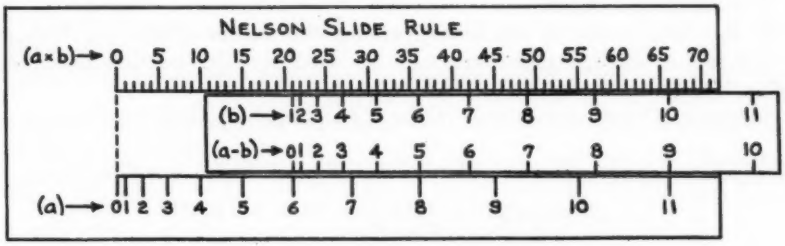

FIG. 1

The operation of the Nelson slide rule can be justified as follows:

Triangular $a=1+\cdots+a$ is an arithmetical progression whose sum is readily shown to be $(a/2)(1+a)=(\frac{1}{2})(a+a^2)$.

Triangular $a-b=1+\cdots+(a-b)=(\frac{1}{2})[(a-b)+(a-b)^2]$.
$=(\frac{1}{2})(a-b+a^2-2ab+b^2)$.

Triangular $b-1=1+\cdots+(b-1)=(\frac{1}{2})[(b-1)+(b-1)^2]$.
$=(\frac{1}{2})(b-1+b^2-2b+1)$.

Then: Triangular a − triangular $a-b$ + triangular $b-1=$
$=(\frac{1}{2})(a+a^2-a+b-a^2+2ab-b^2+b-1+b^2-2b+1)=ab$.

NONLINEAR RECURRENCE RELATIONS FOR CLASSICAL ORTHOGONAL POLYNOMIALS

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1. In recent notes, various authors (*e.g.*, [1], [3]) have considered certain nonlinear recurrence formulas which characterize ultraspherical, Laguerre, and Hermite polynomials. It is the purpose of this note to indicate briefly a method of deriving such formulas for the classical orthogonal polynomials in general. In particular, one such formula is given for the Jacobi polynomials, which have not previously been considered.

The derivations are based on the differential equation [2, 10.7(1)]

$$X\frac{d^2y}{dx^2}+K_1p_1(x)\frac{dy}{dx}+\lambda_n y=0, \qquad y=p_n(x) \tag{1}$$

and Tricomi's differentiation formula [2, 10.7(4)]

$$X\frac{dp_n(x)}{dx} = (\alpha_n + nX''x/2)p_n(x) + \beta_n p_{n-1}(x), \tag{2}$$

which are satisfied by the classical orthogonal polynomials. Here, $X=1$, x, or $1-x^2$ as $\{p_n(x)\}$ denotes the Hermite, Laguerre, or Jacobi polynomials, respectively, and K_1, λ_n, α_n, β_n are constants depending on $\{p_n(x)\}$. For the precise definition of these constants, we refer to [2, Sec. 10.7] whose notation will be followed in this paper.

2. We will derive a relation involving the combination, $(p_n')^2 - p_n p_n''$, $p_n = p_n(x)$. A similar procedure can be used for the combination, $(p_n')^2 - p_{n+1}'(x)p_{n-1}'(x)$.

Squaring (2) and multiplying (1) by Xp_n and subtracting, we obtain

$$X^2[(p_n')^2 - p_n p_n''] = [(\alpha_n + nX''x/2)^2 + \lambda_n X]p_n^2 + 2\beta_n(\alpha_n + nX''x/2)p_n p_{n-1} + \beta_n^2 p_{n-1}^2 + K_1 p_1(Xp_n').$$

Using (2) to eliminate Xp_n' from the right side of this relation yields

$$X^2[(p_n')^2 - p_n p_n''] = U_n p_n^2 + \beta_n[2(\alpha_n + nX''x/2) + K_1 p_1]p_n p_{n-1} + \beta_n^2 p_{n-1}^2, \tag{3}$$

where

$$U_n = U_n(x) = (\alpha_n + nX''x/2)^2 + K_1 p_1(\alpha_n + nX''x/2) + \lambda_n X. \tag{4}$$

From the standard recurrence relation [2, 10.3(7)], we have

$$\beta_n^2 p_{n-1}^2 = \beta_n^2[(A_n x + B_n)p_n - p_{n+1}]p_{n-1}/C_n$$

so that (3) becomes finally

$$X^2[(p_n')^2 - p_n p_n''] = U_n p_n^2 - (\beta_n^2/C_n)p_{n-1}p_{n+1} + \beta_n V_n p_n p_{n-1}, \tag{5}$$

where

$$V_n = V_n(x) = 2(\alpha_n + nX''x/2) + \beta_n(A_n x + B_n)/C_n + K_1 p_1(x). \tag{6}$$

If $p_n(x) = H_n(x)$, $L_n^{\alpha}(x)$, or $C_n^{\lambda}(x)$, (5) reduces to formulas considered by Webster [3, (5), (6) and Theorem 3].* For the Jacobi case, $p_n(x) = P_n^{(\alpha,\beta)}(x)$, the coefficients in (5) become:

$$U_n = n\left[x^2 - \frac{2(\alpha-\beta)}{2n+\alpha+\beta}x + n + \alpha + \beta + 1 - \frac{(\alpha^2-\beta^2)(n+\alpha+\beta)}{(2n+\alpha+\beta)^2}\right],$$

* The derivation of (2) uses the orthogonality of $\{p_n(x)\}$. Hence (5) as derived requires $\alpha > -1$, $2\lambda > -1$, while the formulas of Webster only require α, $2(\lambda-1) \neq -m$ $(m = 1, 2, 3, \cdots)$.

$$\beta_n^2/C_n = \frac{4(n+1)(n+\alpha)(n+\beta)(n+\alpha+\beta+1)}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)},$$

$$\beta_n V_n = -\frac{2(n+\alpha)(n+\beta)}{2n+\alpha+\beta}\left[x + \frac{\alpha^2-\beta^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}\right].$$

Since $\beta_n \neq 0$, (5) determines p_{n+1} uniquely in terms of p_{n-1} and p_n, hence can be used to characterize the Jacobi polynomials.

References

1. W. A. Al-Salam, On a characterization of some orthogonal functions, this MONTHLY, vol. 64, 1957, pp. 29–32.
2. The Bateman Manuscript Project, Higher Transcendental Functions, vol. 2, New York, 1953.
3. M. S. Webster, Non-linear recurrence relations for certain classical functions, this MONTHLY, vol. 64, 1957, pp. 249–252.

NOTE ON ALTERNATING TANNERY SERIES

R. B. LEIPNIK, University of Washington

We call a limit of the form $\lim_{n\to\infty}\sum_{k=1}^{q_n} a_{k,n}$ a Tannery series. Such series often arise in applications. If $\lim_{n\to\infty} q_n = \infty$ and $\lim_{n\to\infty} a_{k,n} = a_k$, it is natural to expect that

$$\lim_{n\to\infty}\sum_{k=1}^{q_n} a_{k,n} = \sum_{k=1}^{\infty} a_k$$

whenever the latter converges.

From an example given by Bromwich* (p. 137), we see that this need not happen. Let $a_{k,n} = \log\{1+(k/n^\alpha)\}$, $q_n = n$, $\alpha > 3/2$. Clearly, $a_k = \lim_{n\to\infty} a_{k,n} = \log 1 = 0$ so that $\sum_{k=1}^{\infty} a_k = 0$. However, we have

$$\frac{k}{n^\alpha} - \frac{1}{2}\frac{k^2}{n^{2\alpha}} \leqq \log\left(1+\frac{k}{n^\alpha}\right) \leqq \frac{k}{n^\alpha}$$

so that

$$\frac{1}{n^\alpha}\frac{n(n+1)}{2} - \frac{1}{2n^{2\alpha}}\frac{n(n+1)(2n+1)}{6} \leqq \sum_{k=1}^{q_n}\log\left(1+\frac{k}{n^\alpha}\right) \leqq \frac{1}{n^\alpha}\frac{n(n+1)}{2}.$$

Since $\alpha > 3/2$, we find $\lim_{n\to\infty} n(n+1)(2n+1)/n^{2\alpha} = 0$. Hence

$$\lim_{n\to\infty}\sum_{k=1}^{q_n} a_{k,n} = \lim_{n\to\infty}\frac{n^{1-\alpha}(n+1)}{2} = \begin{cases} 0, & \alpha > 2; \\ 1/2, & \alpha = 2; \\ \infty, & 3/2 < \alpha < 2. \end{cases}$$

* T. J. I'A. Bromwich, An Introduction to the Theory of Infinite Series, 2nd ed., rev., London, 1955.

Thus the expected result materializes only for $\alpha>2$, the limit is different but finite for $\alpha=2$, and infinite if $3/2<\alpha<2$. (Bromwich considers $\alpha=2$ only.)

Tannery proved the following useful theorem (Bromwich, p. 136) somewhat analogous to the Lebesgue domination theorem for integrals.

THEOREM 1. *If*

$$\sum_{k=1}^{\infty} \sup_n |a_{k,n}| < \infty, \qquad \lim_{n\to\infty} q_n = \infty, \qquad \lim_{n\to\infty} a_{k,n} = a_k,$$

then

$$\lim_{n\to\infty} \sum_{k=1}^{q_n} a_{k,n} = \sum_{k=1}^{\infty} a_k.$$

Unfortunately, this is far from being a necessary condition for the interchange of these limiting operations. In a probability problem, the author encountered a Tannery series, not covered by Theorem 1, whose terms alternate in a way reminiscent of Leibniz's theorem on alternating series. A result that takes care of this situation is

THEOREM 2. *If* $\{q_n\}$ *is an integer sequence with* $\lim_{n\to\infty} q_n=\infty$; $\{f_{k,n}\}$ *is a nonnegative double sequence such that for some* n_0, $f_{k,n}\geqq f_{k+1,n}$ *whenever* $1\leqq k\leqq q_n$ *and* $n>n_0$; $\lim_{n\to\infty} f_{k,n}=f_k$ *is finite for all* k; *and* $\lim_{k\to\infty} f_k=0$; *then*

$$\lim_{n\to\infty} \sum_{k=1}^{q_n} (-1)^{k-1} f_{n,k} = \sum_{k=1}^{\infty} (-1)^{k-1} f_k.$$

Proof. The same reasoning used in proving the ordinary Leibniz theorem shows that, for $n>n_0$, $q_n\geqq 2s+1$, $s\geqq 1$, the inequalities

$$\sum_{k=1}^{2s} (-1)^{k-1} f_{k,n} \leqq \sum_{k=1}^{q_n} (-1)^{k-1} f_{k,n} \leqq \sum_{k=1}^{2s+1} (-1)^{k-1} f_{k,n} \tag{1}$$

follow from the monotone character of $\{f_{k,n}\}$. Taking limits superior and inferior in (1) as $n\to\infty$ and using the existence of $\lim_{n\to\infty} f_{k,n}=f_k$, we find

$$\sum_{k=1}^{2s} (-1)^{k-1} f_k \leqq \underline{\lim_{n\to\infty}} \sum_{k=1}^{q_n} (-1)^{k-1} f_{k,n} \leqq \sum_{k=1}^{2s+1} (-1)^{k-1} f_k \tag{2}$$

for $s\geqq 1$. Clearly, $f_k=\lim_n f_{k,n}\geqq f_{k+1}=\lim_n f_{k+1,n}$, so that from the assumption $\lim_k f_k=0$ we find that $\sum_{k=1}^{\infty}(-1)^{k-1}f_k$ converges by the ordinary Leibniz theorem. Taking limits as $s\to\infty$ in (2), we have

$$\sum_{k=1}^{\infty} (-1)^{k-1} f_k \leqq \underline{\lim_{n\to\infty}} \sum_{k=1}^{q_n} (-1)^{k-1} f_{k,n} \leqq \sum_{k=1}^{\infty} (-1)^{k-1} f_k. \tag{3}$$

Hence the conclusion of the theorem follows.

CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

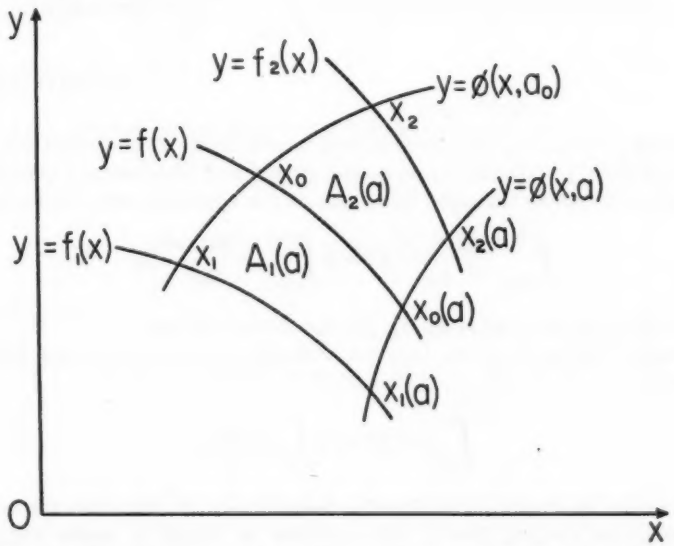
All material for this department should be sent to C. O. Oakley, Department of Mathematics, Haverford College, Haverford, Pa.

ON THE BANKS OF THE NILE

H. J. HAMILTON, Pomona College

A number of years ago someone—I forget who—posed the problem of finding that common boundary between two adjacent pieces of property along the Nile for which, however high the River might flood, the ratio between the inundated portions of these properties would remain the same. Under certain simplifying assumptions, the solution is easily obtained, as we shall now show.

We suppose first that it is possible to orient axes so that the representation in the figure is achieved. Here the curve $y=\phi(x, a_0)$ is the low-water edge of the River, $y=\phi(x, a)$ is the edge at flood time (the variability of which is indicated by the parameter a), $y=f(x)$ is the desired common boundary, and $y=f_1(x)$ and $y=f_2(x)$ are the remaining relevant parts of the boundaries of the pieces of property (the inundated parts of which we call $A_1(a)$ and $A_2(a)$, respectively). The labels attached to the points of intersection among these curves which are shown are their abscissas.



Next, we suppose that, on sufficient intervals of the x- and a-axes and in sufficient regions of the $x-a$ plane—as case may be—$f_1(x)$ and $f_2(x)$ are con-

tinuous; $x_0(a)$, $x_1(a)$, and $x_2(a)$ are differentiable; and $\phi(x, a)$ and $\partial\phi(x, a)/\partial a$ are continuous.

Then, *if, for a given fixed ratio r, the equation $A_1(a) = rA_2(a)$ has a solution $f(x)$ which is continuous on a sufficient interval, this solution is given by $y = \phi(x, a)$, where a satisfies the equation*

$$\int_{x_1(a)}^{x} \frac{\partial\phi(t, a)}{\partial a}\,dt = r\int_{x}^{x_2(a)} \frac{\partial\phi(t, a)}{\partial a}\,dt.$$

To prove this, we now write the equation $A_1(a) = rA_2(a)$ in terms of integrals,

$$\int_{x_1}^{x_0} \phi(x, a_0)dx + \int_{x_0}^{x_0(a)} f(x)dx - \int_{x_1}^{x_1(a)} f_1(x)dx - \int_{x_1(a)}^{x_0(a)} \phi(x, a)dx$$
$$= r\left\{\int_{x_0}^{x_2} \phi(x, a_0)dx + \int_{x_2}^{x_2(a)} f_2(x)dx - \int_{x_0}^{x_0(a)} f(x)dx - \int_{x_0(a)}^{x_2(a)} \phi(x, a)dx\right\},$$

and differentiate with respect to a:

$$f[x_0(a)]x_0'(a) - f_1[x_1(a)]x_1'(a) - \int_{x_1(a)}^{x_0(a)} \frac{\partial\phi(x, a)}{\partial a}\,dx - \phi[x_0(a), a]x_0'(a)$$
$$+ \phi[x_1(a), a]x_1'(a)$$
$$= r\left\{f_2[x_2(a)]x_2'(a) - f[x_0(a)]x_0'(a) - \int_{x_0(a)}^{x_2(a)} \frac{\partial\phi(x, a)}{\partial a}\,dx - \phi[x_2(a), a]x_2'(a)\right.$$
$$\left. + \phi[x_0(a), a]x_0'(a)\right\}.$$

Since $f[x_0(a)] = \phi[x_0(a), a]$, the first and fourth terms in this equation cancel; and since $f_1[x_1(a)] = \phi[x_1(a), a]$, so do the second and fifth terms. Corresponding cancellations occur on the right hand side of the equation, and we are left with

$$\int_{x_1(a)}^{x_0(a)} \frac{\partial\phi(x, a)}{\partial a}\,dx = r\int_{x_0(a)}^{x_2(a)} \frac{\partial\phi(x, a)}{\partial a}\,dx.$$

Since, finally, $f[x_0(a)] = \phi[x_0(a), a]$, the conclusion follows.

Example. Taking $f_1(x) = 0$, $f_2(x) = x$, and $\phi(x, a) = 2x - a$, we are led to the equation

$$\int_{a/2}^{x} (-1)dt = r\int_{x}^{a} (-1)dt,$$

whose solution is $a = 2(r+1)x/(2r+1)$. Substitution of this into the equation $y = 2x - a$ gives $y = 2rx/(2r+1)$, the validity of which is easily established geometrically.

(We would welcome any information you may have on this problem. Ed.)

NOTE ON THE EULER-MACLAURIN FORMULA

W. D. MUNRO, University of Minnesota*

When the subject of numerical integration is introduced into a first course in calculus it usually begins with the trapezoidal rule and ends with Simpson's rule. The doubly useful Euler-Maclaurin formula is generally ignored, possibly because of the difficulty at this level of presenting any of the standard derivations. A formal and heuristically appealing derivation—which seems never to appear in the texts—may be based on nothing more than the Taylor formula or series. Any number of terms can be obtained.

We seek an approximate value for $\int_a^{a+h} f(x)dx$, where h is "small." Let $F(x)$ be such that $F'(x)=f(x)$. Then

$$\int_a^{a+h} f(x)dx = F(a+h) - F(a).$$

By Taylor's expansion about a,

$$F(a+h) = F(a) + f(a)h + \frac{f'(a)h^2}{2!} + \cdots,$$

and

$$F(a+h) - F(a) = f(a)h + \frac{f'(a)h^2}{2!} + \frac{f''(a)h^2}{3!} + \cdots.$$

By expansion about $a+h$,

$$F(a) = F(a+h) - f(a+h)h + \frac{f'(a+h)h^2}{2!} + \cdots$$

and

$$F(a+h) - F(a) = f(a+h)h - \frac{f'(a+h)h^2}{2!} + \cdots.$$

Addition yields

$$\int_a^{a+h} f(x)dx = \frac{1}{2}\left\{h[f(a+h)+f(a)] - \frac{h^2}{2!}[f'(a+h) - f'(a)] + \frac{h^3}{3!}[f''(a+h)+f''(a)] + \cdots\right\}. \tag{1}$$

* The preparation of this paper was sponsored in part by the Office of Naval Research, U. S. Navy, while the author was a visiting staff member of Numerical Analysis Research, University of California, Los Angeles. Reproduction in whole or in part is permitted for any purpose of the United States Government.

Omission of all terms involving h^2 and beyond gives the trapezoidal rule. We attempt to improve this by including terms in h^2 and h^3. A second Taylor expansion about each point gives

$$f(a+h)-f(a)=f'(a)h+\frac{f''(a)h^2}{2!}+\frac{f'''(a)h^3}{3!}+\cdots$$

$$=f'(a+h)h-\frac{f''(a+h)h^2}{2!}+\frac{f'''(a+h)h^3}{3!}+\cdots$$

and subtraction yields

$$-[f'(a+h)-f'(a)]+\frac{h}{2!}[f''(a+h)+f''(a)]$$

$$-\frac{h^2}{3!}[f'''(a+h)-f'''(a)]+\cdots=0. \tag{2}$$

Using only two terms we have approximately,

$$f''(a+h)+f''(a)=\frac{2}{h}[f'(a+h)-f'(a)].$$

Substitution in (1) gives

$$\int_a^{a+h}f(x)dx=\frac{h}{2}[f(a+h)+f(a)]-\frac{h^2}{12}[f'(a+h)-f'(a)]$$

$$+\text{(terms in } h^4 \text{ and beyond)},$$

which agrees with the first two terms of the Euler-Maclaurin formula.

If (2) is used with obvious extension

$$f''''(a+h)+f''''(a)=\frac{2}{h}[f'''(a+h)-f'''(a)],$$

approximately,

$$\int_a^{a+h}f(x)dx=\frac{h}{2}[f(a+h)+f(a)]-\frac{h^2}{12}[f'(a+h)-f'(a)]$$

$$+\frac{h^4}{720}[f'''(a+h)-f'''(a)]+\text{(terms in } h^6 \text{ and beyond)}.$$

Application to an interval (a, b) with $x_k=a+kh$, $h=(b-a)/n$ and $k=0, \cdots, n$ gives the standard form

$$\int_a^b f(x)dx = h\sum_{k=0}^{n} f(x_k) - \frac{h}{2}[f(b) + f(a)] - \frac{h^2}{12}[f'(b) - f'(a)] + \frac{h^4}{720}[f'''(b) - f'''(a)] + \cdots.$$

This is especially useful as a summation formula in the form

$$\sum_{k=0}^{n} f(x_k) = \frac{1}{h}\int_a^b f(x)dx + \frac{1}{2}[f(b) + f(a)] + \frac{h}{12}[f'(b) - f'(a)] + \cdots.$$

For example if terms through the first derivative are used to calculate $\log n! = \sum_{k=1}^{n} \log k$, we have approximately (with $h=1$, $a=1$),

$$\log n! = \sum_{k=1}^{n} \log k = \int_1^n \log x dx + \frac{1}{2}\log n + \frac{1}{12}\left[\frac{1}{n} - 1\right]$$
$$= \left(n + \frac{1}{2}\right)\log n - n + \frac{1}{12n} + \frac{11}{12}.$$

This gives as an approximation to $n!$ the result $e^{11/12}\sqrt{n}(n/e)^n e^{1/12n}$, which compares favorably with the well known Stirling result since $e^{11/12} = .998\sqrt{2\pi}$.

The question naturally arises as to whether a continuation of the procedure will always yield terms in agreement with those of the Euler-Maclaurin formula. Are we really generating the correct coefficients in terms of the Bernoulli numbers or is it mere coincidence for the first few? It is possible (albeit tedious) to establish that the coefficients here obey the correct generating law for the Bernouilli numbers as given, say, by $x/(e^x - 1)$. A simpler device is, however, available. Let the coefficients in the two series be given by

$$\int_a^{a+h} f(x)dx = \frac{h}{2}[f(a+h) + f(a)] - \sum_{k=1}^{\infty} A_{2k}h^{2k}[f^{(2k-1)}(a+h) - f^{(2k-1)}(a)],$$

$$\int_a^{a+h} f(x)dx = \frac{h}{2}[f(a+h) + f(a)] - \sum_{k=1}^{\infty} B_{2k}h^{2k}[f^{(2k-1)}(a+h) - f^{(2k-1)}(a)].$$

Either formula is exact for a polynomial, since the series terminate. Let $f(x) = x^{2n}$, $h=1$, $a=0$, then

$$\int_0^1 x^{2n}dx = \frac{1}{2n+1} = \frac{1}{2} - [A_2(2n) + A_4(2n)(2n-1)(2n-2) + \cdots + A_{2n}(2n)!]$$
$$= \frac{1}{2} - [B_2(2n) + B_4(2n)(2n-1)(2n-2) + \cdots + B_{2n}(2n)!]$$

and a simple induction establishes $A_{2n} = B_{2n}$, all n. This also gives, simply, the relation connecting the A_{2k}, $k = 1, \cdots, n$.

DIOPHANTINE PROBLEMS HAVING NO SOLUTION

J. G. CAMPBELL, Albany, Kentucky

In this note a criterion is given for determining that certain diophantine equations have no solution in rational integers.

Given the equation

$$A_1x_1^{r_1(p-1)} + A_2x_2^{r_2(p-1)} + \cdots + A_nx_n^{r_n(p-1)} = B, \tag{1}$$

where the A's are rational integers, positive, negative or zero and B is a positive integer. Let the r's be positive integers and let p be a rational prime. We shall prove the following theorem.

THEOREM. *Let* S_i, $i=1, \cdots, 2^n-1$ *represent all possible sums of the* A*'s taken* $1, \cdots, n$ *at a time and let* $S_0=0$. *Then* (1) *has no solution in rational integers if*

$$B - S_i \not\equiv 0 \pmod{p} \quad \text{for} \quad i = 0, 1, \cdots, 2^n - 1. \tag{2}$$

Proof. Assume that $x_i=a_i$, for $i=1, \cdots, n$, is a solution of (1) where each a_i is a rational integer. For notational convenience assume that the terms of (1) are so arranged that for some $0 \leqq k \leqq n$, a_i is divisible by p if $i<k$ and prime to p otherwise. We have then, for $i \leqq k$,

$$A_ia_i^{r_i(p-1)} \equiv 0 \pmod{p},$$

and for the remaining terms, if any,

$$A_ja_j^{r_j(p-1)} - A_j \equiv 0 \pmod{p}.$$

Adding these congruences yields,

$$B - S_j \equiv 0 \pmod{p}$$

for some S_j. This contradicts condition (2) and proves the theorem.

The following example is used to illustrate the theorem. Consider the equation

$$x^4 + y^4 + z^4 = 7359. \tag{3}$$

In this example the S_i are 0, 1, 1, 1, 2, 3, 2, 3, so that $B-S_i$ takes on the values 7359, 7358, 7357, 7356. None of these numbers is divisible by 5, hence (3) has no solution in rational integers. Note that an equation whose left side is that of (3) has no solution in rational integers if the number B on the right side ends in either four or nine.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within the three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1306. *Proposed by J. Gallego-Diaz, Vanderbilt University*

A girl entered a store and bought x flowers for y dollars (x and y integers). When she was about to leave, the clerk said, "If you buy ten more flowers I shall give you all for two dollars, and you will save 80 cents a dozen." Find x and y.

E 1307. *Proposed by Leon Bankoff, Los Angeles, Calif.*

Let P denote the contact of AB with the circle tangent internally to the circumcircle of a triangle ABC and touching the sides AB and AC. If Q is the foot of the perpendicular from P upon AC, show that PQ is equal to the diameter of the incircle of triangle ABC.

E 1308. *Proposed by I. S. Gál, Cornell University*

Let A_1, A_2, A_3, A_4 be the vertices of a square and let P be an arbitrary point in the plane. Show that

$$\sum_{k=1}^{4} A_kP \geqq (1+\sqrt{2}) \max_{(k)} A_kP + \min_{(k)} A_kP$$

and determine the points P for which equality takes place.

E 1309. *Proposed by Martin Gardner, New York, N. Y.*

Dissect a regular pentagram (five pointed star) into no more than nine pieces which can be reassembled to form a square. Pieces may be turned over.

E 1310. *Proposed by J. L. Brown, Jr., Pennsylvania State University*

Evaluate $\int_0^{\pi} e^{\cos x} \sin nx \sin(\sin x)dx$, where n is a positive integer.

SOLUTIONS

A Well-concealed Endpoint Maximum

E 1276 [1957, 504]. *Proposed by C. S. Ogilvy, Hamilton College*

Find the greatest right circular cylinder coaxal with and inscribed in the solid formed by rotating around the y-axis the area bounded by the two axes, the parabola $y=9x^2-28x+24$, and the parabola's minimum ordinate.

Solution by D. C. B. Marsh, Colorado School of Mines. The volume of any inscribed coaxal right circular cylinder for the given solid is

$$V(x) = \pi x^2(9x^2 - 28x + 24) \quad \text{for} \quad 0 \leqq x \leqq 14/9.$$

Since $V(x)$ is continuous and differentiable over the closed interval, it may have extrema only where $V'(x)=0$ or at the endpoints. $V'(x)=12\pi x(x-1)(3x-4)$. There are relative minima at $x=0$ and $x=4/3$, and relative maxima at $x=1$ and $x=14/9$. By inspection of the latter, the absolute maximum is seen to occur at the endpoint for the cylinder with altitude 20/9, radius 14/9, and volume $3920\,\pi/729$.

Also solved by H. L. Baldwin, Leon Bankoff, A. P. Boblétt, A. L. Epstein, Michael Goldberg, A. G. Grace, Jr., Cornelius Groenewoud, A. R. Hyde, J. D. E. Konhauser, C. F. Pinzka, David Zeitlin, and the proposer.

Extreme Parameters in an Inequality

E 1277 [1957, 504]. *Proposed by Alexander Oppenheim, University of Malaya, Singapore, Malaya*

For each $p>0$ there is a greatest q and a least r such that

$$\frac{q \sin x}{1 + p \cos x} \leqq x \leqq \frac{r \sin x}{1 + p \cos x}$$

for $0 \leqq x \leqq \pi/2$. Determine q and r as functions of p.

Solution by W. B. Carver, Cornell University. For $x=0$ the required relation holds for all values of q and r. For $0<x\leqq\pi/2$ and for positive p we may multiply by the positive quantity $(1+p\cos x)/x$, and the required relation takes the form

$$(q \sin x)/x \leqq 1 + p \cos x \leqq (r \sin x)/x.$$

With p fixed we consider two curves, the fixed curve C,

$$y = 1 + p \cos x,$$

and the movable curve C_s,

$$y = (s \sin x)/x \text{ for } x \neq 0, \qquad y = s \text{ for } x = 0,$$

where s is a variable parameter. We are interested only in the arcs of these curves in the interval $0 \leqq x \leqq \pi/2$.

The curve C slopes downward from $y=p+1$ at $x=0$ to $y=1$ at $x=\pi/2$; and the curve C_s also slopes downward from $y=s$ at $x=0$ to $y=2s/\pi$ at $x=\pi/2$. The whole arc of the curve C_s moves up or down according as the parameter s is increased or decreased, and the problem is to find the limits q and r of the range of values of s for which the arc of C_s has any points in common with the arc of C.

The curve C and the curve C_s when $s=p+1$ (we may say the curve C_{p+1}) are tangent at $x=0$ with the common tangent having a slope 0. An examination of several terms in the expansions of $1+p\cos x$ and $[(p+1)\sin x]/x$ in powers of x shows that if $p \leqq 1/2$ the curve C_{p+1} lies below the curve C when x is near zero, and if $p>1/2$ the curve C_{p+1} is above C for x near zero.

Similarly, the curve $C_{\pi/2}$ cuts the curve C at $x=\pi/2$; and by expanding

$$1 + p \cos(\pi/2 - h) \quad \text{and} \quad [\pi \sin(\pi/2 - h)]/2(\pi/2 - h)$$

in powers of h one sees that the curve $C_{\pi/2}$ is above or below the curve C for x near $\pi/2$ according as $p<2/\pi$ or $p\geqq 2/\pi$. For $p=\pi/2-1$ the curves C_{p+1} and $C_{\pi/2}$ are, of course, the same.

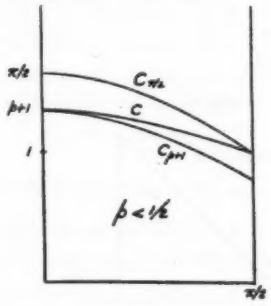


FIG. 1

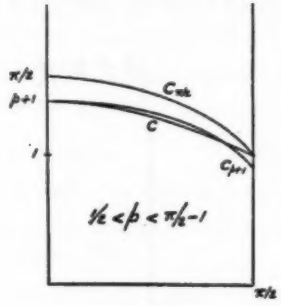


FIG. 2

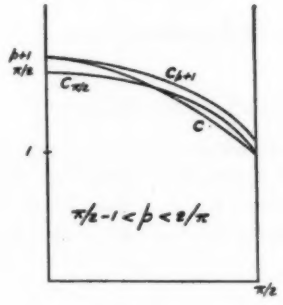


FIG. 3

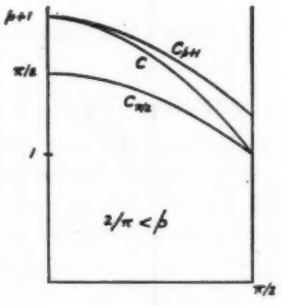


FIG. 4

The relative positions of the curves C, C_{p+1}, and $C_{\pi/2}$ for various values of p are shown in Figures 1–4. One sees that the least value of r required by the problem is

$$r = \pi/2 \quad \text{when} \quad p \leqq \pi/2 - 1,$$
$$r = p + 1 \quad \text{when} \quad p \geqq \pi/2 - 1.$$

Also the required greatest value of q is

$$q = p + 1 \quad \text{when} \quad p \leqq 1/2,$$
$$q = \pi/2 \quad \text{when} \quad p \geqq 2/\pi.$$

The difficult part of the problem is to find the greatest value q for values of p

in the interval

$$1/2 < p \leqq \pi/2 - 1 \quad \text{and} \quad \pi/2 - 1 \leqq p < 2/\pi.$$

Figures 2 and 3 indicate that when p has a value in the interval $1/2 < p < 2/\pi$ the required greatest value of q is that value s (other than $s = p+1$) for which the curve C_s has a point of tangency with the curve C. For such tangency we must have

$$(1) \qquad \begin{cases} 1 + p \cos x = (s \sin x)/x, \\ -p \sin x = s(x \cos x - \sin x)/x^2. \end{cases}$$

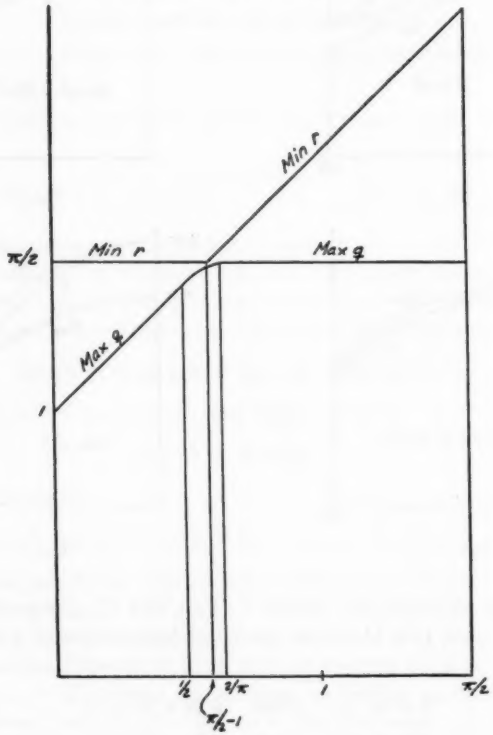


FIG. 5

Theoretically, the elimination of x between these two equations would give a relation between s and p from which we might express s (the required greatest value of q) as a function of p; but obviously this function of p will not have any simple closed form. If we solve equations (1) for s and p in terms of x we have

$$(2)\qquad \begin{cases} s = (x^2 \sin x)/(x - \sin x \cos x), \\ p = (\sin x - x \cos x)/(x - \sin x \cos x). \end{cases}$$

By then assigning numerical values of x we can calculate pairs of corresponding values of s and p, with p in the range $1/2 < p < 2/\pi$. For $x = \pi/3$, for example, we have $p = 0.5575$, $s = 1.546$. The required least r and greatest q are shown graphically as functions of p in Figure 5. Equations (2) may be regarded as parametric equations for the part of the curve Max q in the interval $1/2 < p < 2/\pi$.

Also solved by W. H. Laubach and D. C. B. Marsh (jointly), and the proposer. Late solutions by Julian Braun and Ya'akov Shima.

A Property of the Mystic Hexagram

E 1278 [1957, 504]. *Proposed by V. F. Ivanoff, San Carlos, Calif.*

If a hexagon $ABCDEF$ is inscribed in a conic, and if P is any point either on the conic or on the Pascal line of the hexagon, then

$$(\sin APB \sin CPD \sin EPF)/(\sin BPC \sin DPE \sin FBA) = \text{constant}.$$

Solution by the proposer. If P is on the conic, then

$$(\sin APB \sin CPD)/(\sin BPC \sin APD) = \text{constant}$$

and

$$(\sin DPE \sin FPA)/(\sin EPF \sin DPA) = \text{constant},$$

and the desired result follows by division. We have to use a different approach, however, to establish the result for the more general position of P.

Let $a_1 = 0$, $a_2 = 0$, $a_3 = 0$ represent equations of the lines AB, CD, EF, and let $b_1 = 0$, $b_2 = 0$, $b_3 = 0$ represent equations of the lines DE, FA, BC. Then

$$(1)\qquad a_1 a_2 a_3 = \lambda b_1 b_2 b_3,$$

where λ is an arbitrary constant, is the equation of a cubic passing through the vertices of the hexagon and through the three Pascal points of the Pascal line of the hexagon. (The three Pascal points are the points of intersection of the pairs of lines AB, DE; CD, FA; EF, BC.) By a proper choice of λ we obtain the degenerate cubic composed of the conic and the Pascal line. (To get this cubic we set the condition that the cubic (1) pass through a fourth point of the Pascal line, and from this condition find the value of λ.) Now the equation $a_1 = 0$ may be taken in the form

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0,$$

where (x_1, y_1), (x_2, y_2) are the coordinates of points A and B. If (x, y) are the coordinates of point P, then the determinant in the above equation is equal to

$PA \cdot PB \sin APB$, both expressions being equal to twice the signed area of triangle PAB. In this way $a_1, a_2, a_3, b_1, b_2, b_3$ can be made to represent twice the signed areas of triangles PAB, PCD, PEF, PDE, PFA, PBC, respectively, whence, from the equation of the degenerate cubic, we have

$$\frac{PA \cdot PB \sin APB \; PC \cdot PD \sin CPD \; PE \cdot PF \sin EPF}{PB \cdot PC \sin BPC \; PD \cdot PE \sin DPE \; PF \cdot PA \sin FPA} = \text{constant}.$$

Cancellation now yields the desired result.

Cutting an n×n×n Cube into n^3 Unit Cubes

E 1279 [1957, 504]. *Proposed by L. R. Ford, Jr., and D. R. Fulkerson, The RAND Corporation, Santa Monica, Calif.*

In the February 1957 issue of *Scientific American* the following problem appeared: "A carpenter, working with a buzz saw, wishes to cut a wooden cube, three inches on a side, into 27 one-inch cubes. He can do this easily by making six cuts through the cube, keeping the pieces together in the cube shape. Can he reduce the number of necessary cuts by rearranging the pieces after each cut?" Generalize this to find the number of cuts necessary to dissect an $n \times n \times n$ cube into n^3 one-inch cubes.

Solution by D. C. B. Marsh, Colorado School of Mines. We first note that only one dimension of any one piece may be divided at each cut, and that cuts which come as near as possible to bisecting each piece yield the optimum. Thus, if pieces are piled after each cutting, an n-unit length will be divided into unit segments by k cuts, where k is defined by $2^k \geqq n > 2^{k-1}$, and unit cubes will be attained with $3k$ cuts.

Also solved by Merrill Barnebey, P. L. Chessin, W. J. Cody, Michael Goldberg, Frank Hawthorne, Werner Held, James Jordan, H. G. Loomis, Leo Moser, C. S. Ogilvy, and the proposers. Late solution by B. D. Roberts.

It was pointed out that this problem is a special case of Leo Moser's Problem 102, *Mathematics Magazine*, vol. 25, no. 4, March–April 1952, p. 219. There it is more generally shown that the least number of planar cuts required to cut an $a \times b \times c$ block into abc unit cubes, with piling permitted, is

$$3 + [\log_2 (a - 1)] + [\log_2 (b - 1)] + [\log_2 (c - 1)].$$

The problem can also be further generalized to an r-dimensional box of dimensions $a_1, a_2, \cdots, a_r$; here the minimal number of cuts is given by

$$r + \sum_{i=1}^{r} [\log_2 (a_i - 1)].$$

Three Squares in Arithmetic Progression

E 1280 [1957, 505]. *Proposed by V. K. Narayanan, Puthenchanthai, Trivandrum, South India*

If $x=24$, then $x+1$ and $2x+1$ are perfect squares; if $x=40$, then $2x+1$ and $3x+1$ are perfect squares; if $x=8$, then $x+1$ and $3x+1$ are perfect squares. Is there a positive integer x such that all three $x+1$, $2x+1$, $3x+1$ are perfect squares?

Solution by Leonard Carlitz, Duke University. If a positive integer x exists such that $x+1$, $2x+1$, $3x+1$ are all squares, then the numbers 1, $x+1$, $2x+1$, $3x+1$ constitute four squares in arithmetic progression. This has been shown to be impossible. See Dickson's *History of the Theory of Numbers*, vol. 2, p. 440 and p. 635 (ref. 109).

Also solved by Leon Bankoff, D. A. Breault, P. L. Chessin, Edgar Karst, Sam Kravitz D. C. B. Marsh, and Leo Moser. Late solutions by W. J. Blundon and M. S. Klamkin.

Some of the solutions submitted were not complete. Chessin and Moser employed well-known Pellian theory. The proposer pointed out that it is possible to find positive integers n such that $x+n$, $2x+n$, $3x+n$ will all be perfect squares for some positive x. Thus if $n=193$, then we may take $x=2016$, and if $n=241$ we may take $x=720$. The value $n=193$ is the smallest such n.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4778. *Proposed by R. C. Lyness, Preston, England*

Given $f(r)=\alpha^r(\beta-\gamma)+\beta^r(\gamma-\alpha)+\gamma^r(\alpha-\beta)$ in which α, β, γ, are nonzero and distinct. If n is a positive integer and $f(n+1)=0$, prove that

$$f(n+2)f(n) = \alpha^n\beta^n\gamma^n(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)f(-n).$$

4779. *Proposed by Solomon Leader, Rutgers University*

Let $f(x)=|x-k|$ for $k-\frac{1}{2}\leqq x\leqq k+\frac{1}{2}$, where k runs through the integers, and

$$g_n(x) = \sum_{r=0}^{n-1} 10^{-r}f(10^r x).$$

($\operatorname{Lim}_{n\to\infty} g_n(x)$ is Van der Waarden's example of a continuous, nowhere differentiable function.) Let $\|g_n\|$ be the total variation of g_n on the interval (0, 1). Find $\lim_{n\to\infty} n^{-1/2}\|g_n\|$.

4780. *Proposed by M. S. Klamkin and D. J. Newman, AVCO Research and Development, Lawrence, Mass.*

An equiproduct point of a curve is defined to be a point such that the product of the two segments of any chord through the point is constant. (1) Show that if every point inside a curve is equiproduct, the curve must be a circle. (2) What is the maximum number of equiproduct points a noncircular oval can have?

4781. *Proposed by J. L. Massera, Mathematics Institute, Montevideo, Uruguay*

Let $f(x, y, z, \cdots) = a_0(y, z, \cdots)x^n + a_1(y, z, \cdots)x^{n-1} + \cdots + a_n(y, z, \cdots)$, where the a_i are any real functions defined in any region G of the $(y, z, \cdots)$-space. Let $g(y, z, \cdots)$ be any real function defined in G and construct $f^*(x, y, z, \cdots) = f + gf_x + g^2 f_{xx} + \cdots$. Then, if $f \geqq 0$ in the cylinder $K = G \times \{x: -\infty < x < +\infty\}$, we have $f^* \geqq 0$ in K. More precisely, if the a_i do not vanish simultaneously in G, $f^* > 0$ in K except at the points where $f = g = 0$.

4782. *Proposed by V. F. Ivanoff, San Carlos, Calif.*

Given a composite function $F(x) \equiv f[g(x)]$. Denoting the nth derivative of $f(g)$ by $D^n f$, and the derivatives of $g(x)$ by $g', g'', \cdots, g^{(n)}$, show that

$$F^{(n)}(x) = \begin{vmatrix} g' & g'' & g''' & g^{(iv)} & \cdots & g^{(n)} \\ -1 & g'D & 2g''D & 3g'''D & \cdots & \binom{n-1}{1} g^{(n-1)}D \\ 0 & -1 & g'D & 3g''D & \cdots & \binom{n-1}{2} g^{(n-2)}D \\ 0 & 0 & -1 & g'D & \cdots & \binom{n-1}{3} g^{(n-3)}D \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 \cdots -1 & & g'D \end{vmatrix} Df.$$

SOLUTIONS

Separable Compact Hausdorff Space

4733 [1957, 276]. *Proposed by Albert Wilansky, Lehigh University*

Give an example of a compact Hausdorff space which is separable (has a dense sequence) but not completely separable (does not satisfy the second axiom of countability).

Solution by the proposer. The space will be nonmetrizable, obviously. Let X be the half-open interval of real numbers $(0, 1]$; $\overline{X}$ its Cech compactification. Since $\overline{X} \neq X$, $\overline{X}$ is not metrizable (Hewitt). This follows also from the fact that $C(X)$ is not separable, while $C(\overline{X})$ would be if $\overline{X}$ were metric.

Also solved by G. E. Collins, M. K. Fort, Jr., Leonard Gillman, R. C. James, M. J. Mansfield,

Montford Plebstnoch, R. E. Priest, Helmut Salzmann, H. F. Thornton, and L. E. Ward, Jr.,

Editorial Note. A variety of examples were cited by the solvers. See also J. L. Kelley, *General Topology*, Ex. N, p. 104 and Ex. M, p. 164.

A Vanishing Function

4738 [1957, 369]. *Proposed by R. R. Goldberg, Pittsburgh, Pa.*

If, for all positive x, $\sum_{k=1}^{\infty} |F(kx)| < \infty$ and $\sum_{k=1}^{\infty} F(kx) = 0$, then $F(x)$ vanishes identically.

Solution by D. S. Greenstein, University of Michigan. Let $S_n = \sum_{k=1}^{\infty} F(knx)$ and let $p_1, p_2, \cdots$, be the sequence of primes in ascending order. Clearly $S_n = 0$ for any $n \geqq 1$. Consider

$$(1) \qquad \begin{aligned} A_n &= S_1 - \sum_{i=1}^{n} S_{p_i} + \sum_{i<j\leq n} S_{p_i p_j} - \sum_{i<j<k\leq n} S_{p_i p_j p_k} + \cdots + (-1)^n S_{p_1 \cdots p_n} \\ &= \sum_{(k,\, p_1 \cdots p_n)=1} F(kx) = 0. \end{aligned}$$

Now this last sum contains the term $F(x)$ and no other term $F(kx)$ with $k \leqq p_n$. Therefore we have

$$(2) \qquad |F(x)| = |A_n - F(x)| < \sum_{k=p_n+1}^{\infty} |F(kx)|,$$

but by the absolute convergence of S_n the last member of (2) is arbitrarily small for sufficiently large n. Hence $F(x) = 0$.

Also solved by P. T. Bateman, Ward Cheney, D. D. Dix, N. J. Fine, Charles Fox, Joseph Lehner and G. M. Wing, Gideon Peyser, M. F. Smiley, A. D. Ziebur, and the proposer.

Editorial Note. Bateman provides the following example to show that the hypotheses regarding absolute convergence cannot be discarded. Let

$$F(x) = \begin{cases} 0 & \text{if } x \text{ is not an integer,} \\ \lambda(x)x^{-1} & \text{if } x \text{ is an integer,} \end{cases}$$

where $\lambda(x) = (-1)^u$, u being the total number of prime factors of x, when multiple factors are counted multiply. If x is irrational, clearly $\sum_{k=1}^{\infty} F(kx) = 0$. If $x = a/b$, where a and b are coprime positive integers, then

$$\sum_{k=1}^{\infty} F(kx) = \sum_{k=1,\, b|k}^{\infty} F(ka/b) = \sum_{n=1}^{\infty} F(na) = \sum_{n=1}^{\infty} \lambda(na) n^{-1} a^{-1} = \lambda(a) a^{-1} \sum_{n=1}^{\infty} \lambda(n) n^{-1}.$$

Now, since $\lambda(n) = \sum_{d^2|n} \mu(n/d^2)$, clearly

$$\sum_{n\leq N} \lambda(n) n^{-1} = \sum_{md^2\leq N} \mu(m)(md^2)^{-1} = \sum_{d^2\leq N} d^{-2} \sum_{m\leq Nd^{-2}} \mu(m) m^{-1}.$$

Hence

$$\sum_{n=1}^{\infty} \lambda(n) n^{-1} = 0$$

by Landau's theorem that $\sum_{m=1}^{\infty} \mu(m) m^{-1} = 0$ (*cf.*, Hardy, *Divergent Series*, Oxford, 1949, p. 380). Thus $\sum_{k=1}^{\infty} F(kx) = 0$ for any positive x, even though $F(x)$ is not identically zero.

Intersecting Octahedra

4739 [1957, 369]. *Proposed by V. L. Klee, Jr., University of Washington*

Suppose C is a closed convex subset of the Euclidean space E^3 whose boundary is a regular octahedron, and that C_1, C_2, and C_3 are translates of C (*i.e.*, $C_i = C + x_i$ for some $x_i \in E^3$). Then, if each of the intersections $C_1 \cap C_2$, $C_2 \cap C_3$, and $C_3 \cap C_1$ is nonempty, must $C_1 \cap C_2 \cap C_3$ be nonempty?

Solution by Harley Flanders, University of California. The answer is yes. For suppose $C_1 \cap C_2$ and C_3 are disjoint. We note that C_3 is the intersection of the eight closed half-spaces determined by its faces, consequently there must be a plane P which contains a face of C_3 such that $C_1 \cap C_2$ lies entirely in the open half-space determined by P which does not contain C_3. We now choose any plane M perpendicular to P and form the orthogonal projections R_1, R_2, R_3 of C_1, C_2, C_3 on M. The R_i are rhombuses obtained from each other by translation and evidently lie in the impossible configuration in which each two intersect, but $R_1 \cap R_2$ and R_3 are disjoint.

The corresponding result for rhombuses in the plane reduces to a statement about disjoint segments on a line via oblique projection.

The same proof applies when the regular octahedra are merely homothetic.

Editorial Note. Although the proposer had originally an independent argument, he finds the result in a recent paper by Hanner, *Math. Scand.*, vol. 4, 1956, pp. 65–87. If the three-dimensional cartesian space E is metrized by means of the norm $N(x) = |x_1| + |x_2| + |x_3|$ then its unit cell is an octahedron. From the result stated above and another of Hanner's results, it follows that E is 4-hyperconvex but not 5-hyperconvex, in the sense of Aronszajn and Panitchpakdi (*Pacific J. Math.*, vol. 6, 1956, pp. 405–440). This solves part of their Problem 1, p. 437. From another result in Hanner's paper it follows that if a finite-dimensional Banach space is 5-hypercomplex, it is hypercomplex.

Locally Schlicht Mapping

4740 [1957, 370]. *Proposed by R. J. Dickson, Lockheed Aircraft Corporation, Burbank, Calif.*

Is every locally schlicht analytic mapping of the complex plane onto itself a schlicht mapping?

Solution by D. S. Greenstein, University of Michigan. Let $h(z)$ be any non-constant, even, entire function. Consider the odd, transcendental, entire function

$$f(z) = \int_0^z e^{h(t)} dt.$$

Clearly $f(z)$ is locally schlicht since its derivative has no zeros. Since it is odd and transcendental, it must be an onto mapping by virtue of the Picard theorem. Also, by the Picard theorem, it is not schlicht.

Also solved by Harley Flanders and the proposer. Late solution by I. N. Baker.

Complete Metric Space

4741 [1957, 370]. *Proposed by L. A. Rubel, Institute for Advanced Study*

Prove or disprove the statement: If a metric space S is homeomorphic to its completion, then S is complete.

I. *Solution by Richard Arens, University of California, Los Angeles.* Let S^* be the following set in the plane:

$$S^* = \{(x, y): x = 0, 1, 2, \cdots; y = 0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \cdots\}.$$

Let $S = S^* - \{(0, 0)\}$. Then S^* is the completion of S. But the following mapping g, defined on S, rather obviously maps S homeomorphically on S^*:

$$g\left(m, \frac{1}{n}\right) = \left(m - 1, \frac{1}{n+1}\right) \qquad (m > 0),$$

$$g(m, 0) = (m - 1, 0), \qquad g\left(0, \frac{1}{n}\right) = (n - 1, 1).$$

Since S is not complete, the proposed statement is false.

II. *Solution by G. E. Bredon, Harvard University.* Let S be the subset of the plane which consists of the interior of the first quadrant together with the (open) positive x-axis. The completion of this is the closed quadrant and both of these are clearly homeomorphic to the closed half-plane.

Also solved by F. Cunningham, Jr., M. K. Fort, Jr., G. A. Harris, Jr., Meyer Jerison, V. L. Klee, Jr., D. J. Newman, J. R. Schoenfield, and the proposer.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Introduction a L'étude de L'analyse Symbolique. By Maurice Parodi. Gauthier-Villars, Paris, 1957. 246 pp. $10.20.

This is an introductory treatise on the operational calculus using the Laplace transform (the term "symbolic analysis" or "symbolic calculus" for this subject seems not to be used in English). The author has in mind applications of the transform calculus to physics and engineering, and the treatment is pitched for the mathematical technologist rather than for the mathematician. Fortunately this is not one of the books of the kind in which one finds a "proof" that the Heaviside unit step-function has a derivative equal to one at the jump discontinuity. However, one is surprised to find that what is announced as a brief outline of L. Schwartz's theory of distributions turns out to be a standard treat-

ment of the unit impulse (Dirac delta function) by means of the Stieltjes integral (pp. 43–47).

One third of the book consists of a first chapter devoted to the elementary theory of the Laplace transform as a function of a complex variable—in addition to deriving the transforms of many useful functions the text establishes basic properties: analyticity of the transform, behavior at the origin and at infinity, differentiation and integration with respect to a parameter, convolution theorem, Lerch's theorem, the Bromwich-Wagner inversion formula, and the Heaviside expansion theorem. The remainder of the work is devoted to the solution of ordinary and partial differential equations (formally), and brief treatments of the evaluation of definite integrals and the solution of some integral equations. One finds the usual large number of applications to wave propagation, transmission lines, heat flow, elasticity and electric networks. The author's choice of examples well illustrates the range and power of the transform calculus. There are no exercises for the student-reader.

THEODORE HAILPERIN
Lehigh University

Elements of Partial Differential Equations. By Ian N. Sneddon. McGraw-Hill New York, 1957. x+327 pp. $7.50.

Quoting from the author's preface: "The aim of this book is to present the elements of the theory of partial differential equations in a form suitable for the use of students and research workers whose main interest in the subject lies in finding solutions of particular equations, rather than in the general theory."

The above motivation is excellently carried out by introducing the reader to a fairly complete list of techniques to be used in attempting to solve a boundary value problem for a partial differential equation, the techniques being illustrated by examples taken mostly from Laplace's equation, the wave equation, and the diffusion equation.

Each chapter has a fine list of problems which, together with the worked examples, includes many important applications from mathematical physics and engineering. A number of selected results from the general theory is included, with references to more complete treatments. In the same way most proofs are omitted, although those which are supplied are mainly geometrical and fit in well with the intuitive approach used.

The overall impression of the book is that it is somewhat disconnected, due, no doubt, to its goal of providing methods for solving specific problems. It is implicit in the text that the reader has a good grounding in ordinary differential equations including eigenfunction expansions, complex variables and a first course in partial differential equations.

A partial list of topics: Chapter I. Pfaffian differential equations, and integrability conditions. Chapters II, III. Partial differential equations of first and second orders, some theoretical results and special methods of solution. Chapters IV, V, VI. Applications of diverse methods of solving Laplace's equation,

the wave equation and the diffusion equation. Appendix. Systems of surfaces. Answers to odd numbered problems.

GORDON LATTA
Stanford University

The Hypercircle in Mathematical Physics. By J. L. Synge. Cambridge University Press, New York, 1957. xii+424 pp. $13.50.

The aim of this book is to furnish for engineers and mathematical physicists a practical method for obtaining approximate solutions to partial differential equations which is based upon theory available only in pure mathematics. Instead of following the usual procedure of replacing the partial differential equations by a system of equations in finite differences, the author introduces a method by which the solution may be considered as a point in function space, a point of intersection of two linear subspaces orthogonal to each other. Although function space is infinite in dimensionality, use is made of two subspaces each of which has a finite number of points. The points in one subspace are chosen as extremities of vectors satisfying some of the conditions of the problem and other conditions are satisfied by the points (vectors) of the other subspace. Thus, by working in two subspaces, the solution appears as a point lying on the hypercircle of intersection of the subspaces. It is possible to calculate the precision of the result at any stage by finding the radius of the hypercircle. If the radius is zero, an exact solution is obtained.

The concepts of function space employed are developed from the beginning in an extraordinarily lucid manner. Such concepts as orthonormalization, distance in function space as exemplified by the Hilbert, Dirichlet, and Minkowski metrics, minimum property of a normal to a linear n space, and the orthogonality of two linear spaces are carefully explained for later use. A knowledge of elementary calculus and vector analysis should be enough to allow the reader to follow the development. There is a beautiful interplay between geometric and algebraic methods of attack. The author makes use of many figures which serve to convince the reader that a method of proof is often suggested by geometrical intuition. As an aid to understanding many illustrative examples are worked out, and exercises are strategically placed to enable the reader to test his comprehension.

In the one chapter of Part I the geometry of function space without a metric is skillfully portrayed. By the adoption of a positive definite metric throughout the four chapters of Part II enrichment is gained for the geometry of function space by multiplying vectors. In the applications the scalar product or metric is chosen suitable to the boundary value problem dealt with. In Chapter 2 the key to the hypercircle method is revealed after consideration of hyperplanes, hyperspheres, and hypercircles. Chapter 3 considers the Dirichlet problem for a finite domain in the Euclidean plane, the domain of the physical problem being either singly or multiply connected. For a point in function space, a vector field defined over the domain and its boundary is employed. In Chapter 4 a detailed

account is given of the hypercircle method as applied to the torsion of a cylinder, with particular attention to multiply connected cross sections, and the use of pyramid vectors. In Chapter 5, boundary value problems are connected with variational principles, and some examples of mixed boundary-value problems are discussed. In the two final chapters which make up Part III, the geometry of function space with indefinite metric and some applications to vibration problems are considered. Changes in the theory appear because the geometry is now analogous to the Minkowski geometry in the space-time of special relativity. Arithmetical bounds on a solution are no longer available and minimum principles give way to stationary principles.

This book is stimulating for both geometers and analysts, and it must surely be looked into by all who are interested in finding approximate solutions of boundary value problems.

C. E. SPRINGER
University of Oklahoma

Statistical Analysis of Stationary Time Series. By Ulf Grenander and Murray Rosenblatt. Wiley, 1957. 300 pp. $11.00.

This book is a welcome and most useful addition to the Wiley publications in Statistics. It is concerned primarily with random mechanisms producing a process that does not change with t. The probability model considered fits data arising over moderate periods of time in studies of random noise, problems in turbulence and oceanography as well as in some investigations appertaining to meteorology and econometrics.

The monograph gives an excellent and scholarly treatment of this extremely important subject. It fulfills the need for a new approach to time series analysis that differs basically from the many techniques previously used. The authors demonstrate a keen insight into the theory and the various fields of application. They have completed certain research and new results have been obtained. It is also most gratifying to find rigor and clarity in the presentation. The approach here is much more general in scope than that found in most of the earlier work in time series analysis.

There are eight Chapters in the monograph. Chapter I gives the derivation of pertinent results relative to stationary stochastic processes with illustrations taken largely from physical fields. Chapter II discusses several types of linear problems when the covariance function (spectrum) of the stationary process is known. Here one also finds reference to the construction of the optimal linear predictor. In Chapter III we find a practical slant. There is discussed a uniform, very general approach to statistical problems dealing with infinite dimensional models. In Chapter IV the authors discuss various aspects of the problem of estimating the spectral density. Chapter V is devoted to applications of the methods in the study of random noise, turbulence and storm-generated ocean waves. In Chapter VI the authors set up for their results certain statistical tests

of hypotheses or confidence regions. Large sample expressions for bias and the variance of estimates are obtained. Chapter VII gives us a better-than-usual discussion of regression analysis with a discussion of linear unbiased estimates of the coefficients. Chapter VIII contains a very interesting and inspiring assortment of problems on the maxima and zeros of time series and also prediction when the covariance function is estimated from time series. Following Chapter VIII one finds delightful and well chosen problems for each chapter in the monograph. In the reviewer's opinion, these problems with their solutions could well be considered Volume II of the monograph.

This monograph is of use and interest to the physicist, the technologist, the mathematical and applied statistician, the mathematician, and the econometrician. The authors have given the reader a wealth of information in a very clear and concise fashion. In the reviewer's opinion, it will prove to be an excellent text for much-needed courses of instruction in time series analysis as well as to stimulate much needed research in time series analysis. The reader is given ample inspiration for extensions of the analysis. It is important to recognize the nonparametric character of the techniques presented by the authors.

To study and/or read the monograph with understanding, one should have maturity in mathematical statistics and stochastic processes which implies maturity in mathematics.

FRANK M. WEIDA
The George Washington University

BRIEF MENTION

A Guide to Graduate Study. Programs leading to the Ph.D. degree. By Frederic W. Ness, Editor. Association of American Colleges, Washington, D. C., 1957. xi+335 pp. $5.00.

This valuable book should certainly be available to every person who advises senior students planning to continue for graduate study. It contains descriptions of the instructional programs leading to the Ph.D. degree at 135 colleges and universities in the United States, including admission requirements, fees, first year aid programs, fields of study, residence requirements, the number of members of the graduate faculty by department, and enrollments. Physical scientists will be considerably disappointed to note that where 50 or more institutions offer the Ph.D. program in a given field, no specific listing of the school is provided in the "Guide to Fields of Study." Accordingly, chemistry, economics, English, education, chemical engineering, history, mathematics, philosophy, physics and psychology are omitted from this list. They are, of course, listed among the offerings of the individual schools. In examining the book, the reviewer has noted a number of discrepancies between the data given and current actual practice; however, this is no doubt the fault of the persons supplying the data, not of the compiler. The book is a welcome addition.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending News items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

SOUTHERN REGIONAL GRADUATE SUMMER SESSION IN STATISTICS

The fifth Southern Regional Graduate Summer Session in Statistics will be held June 16–July 26, 1958, at Oklahoma State University, Stillwater, Oklahoma. The summer sessions are rotated annually among the following institutions: Virginia Polytechnic Institute, Oklahoma State University, University of Florida and North Carolina State College.

The program may be entered at any session, and consecutive courses will be offered in successive summers. The summer work in statistics may be applied towards residence requirements at any one of the cooperating institutions, as well as certain other institutions, in partial fulfillment of residence requirements for graduate degrees. Each summer session lasts six weeks and the several courses offered carry three semester hours of graduate credit. The summer sessions are designed to carry out a recommendation of the Southern Regional Education Board's Committee on Statistics, on which the four institutions initiating the program are represented.

The sessions will be of particular interest to (1) research and professional workers who want intensive instruction in basic statistical concepts and who wish to learn modern statistical methodology, (2) teachers of elementary statistics courses who want some formal training in modern statistics, (3) prospective candidates for graduate degrees in statistics, (4) graduate students in other fields who desire supporting work in statistics, and (5) professional statisticians who wish to keep informed of advanced specialized theory and methods.

The faculty for the 1958 Summer Session will include the following Visiting Professors: H. O. Hartley, Statistical Laboratory, Iowa State College; T. Federer, Biometrics Unit, Cornell University; J. E. Freund, Department of Mathematics, Arizona State College; A. W. Wortham, Operations Research Department, Texas Instruments, Dallas, Texas.

The local staff includes: Professors C. E. Marshall, F. A. Graybill, R. D. Morrison, J. W. Hamblen, Roy Deal, and J. E. Hoffman.

Of particular interest at this summer session will be the six weekly symposia covering six important areas in statistics. They are: Sampling Survey Designs, Experimental Designs, Nonparametric Statistics, Response Curves and Surfaces, Multiple Comparisons, and High Speed Computing. Discussants will be selected from major contributors to these areas. These invited speakers together with the outstanding summer school staff will cover the respective subjects from three points of view: applications, their mathematical bases, and the problems that lie on the frontier.

Inquiries should be addressed to Dr. C. E. Marshall, Director, Statistical Laboratory, Oklahoma State University, Stillwater, Oklahoma.

NSF INSTITUTES FOR SCIENCE AND MATHEMATICS TEACHERS

The National Science Foundation has announced institutes for high school teachers of science and mathematics for the academic year 1958–59. Inquiries concerning admis-

sion to a particular Institute should be sent directly to the Director of that Institute at the address listed. These Institutes and Directors are as follows:

Chicago, University of: Prof. E. P. Northrop, Eckhart Hall, Box 23, University of Chicago, Chicago 37, Illinois. (Mathematics only).

Colorado, University of: Prof. W. E. Briggs, 318 Hellems Annex, University of Colorado, Boulder, Colorado.

Illinois, University of: Prof. Joseph Landin, Department of Mathematics, University of Illinois, Urbana, Illinois. (Mathematics only).

Iowa State Teachers College: Prof. R. A. Rogers, Department of Science, Iowa State Teachers College, Cedar Falls, Iowa.

Michigan, University of: Prof. F. D. Miller, Department of Astronomy, University of Michigan, Ann Arbor, Michigan.

North Carolina, University of: Prof. E. C. Markham, Department of Chemistry, University of North Carolina, Chapel Hill, North Carolina.

Ohio State University: Prof. J. S. Richardson, 208 Communications Laboratory, Ohio State University, Columbus 10, Ohio.

Oklahoma State University: Prof. J. H. Zant, Department of Mathematics, Oklahoma State University, Stillwater, Oklahoma.

Oregon State College: Prof. S. E. Williamson, Department of Science Education, Oregon State College, Corvallis, Oregon.

Pennsylvania, University of: Dean W. E. Arnold, School of Education, University of Pennsylvania, Philadelphia 4, Pennsylvania.

South Dakota, State University of: Prof. C. M. Vaughn, Department of Zoology, Medical and Science Building, State University of South Dakota, Vermillion, South Dakota.

Syracuse University: Dr. A. T. Collette, 400 Lyman Hall, Syracuse University, Syracuse 10, New York.

Texas, University of: Prof. R. C. Anderson, Chemistry Department, University of Texas, Austin 12, Texas.

Utah, University of: Prof. T. J. Parmley, Room 215, Physical Science Building, University of Utah, Salt Lake City 12, Utah.

Virginia, University of: Prof. J. W. Cole, Jr., Department of Chemistry, University of Virginia, Charlottesville, Virginia.

Washington University: Prof. E. U. Condon, Department of Physics, Washington University, St. Louis 5, Missouri.

Wisconsin, University of: Prof. C. H. Sorum, Chemistry Department, University of Wisconsin, Madison, Wisconsin.

SYMPOSIUM ON NUMERICAL APPROXIMATION

A Symposium on Numerical Approximation sponsored by the Mathematics Research Center, U.S. Army, will be held April 20–23 at the University of Wisconsin. The topics include linear approximation, interpolation, Tchebycheff and other extremal approximations, expansions and algorithms.

One-hour surveys (including a survey of recent Russian literature) and thirty-minute research papers will be presented. There will be opportunity for formal and informal discussion. Approximately twenty speakers will participate. Among these are the following guests from abroad: L. Collatz, L. Fox, Z. Kopal, C. P. Miller, A. Ostrowski, and E. L. Stiefel.

Workers in the field interested in attending the Symposium are urged to write to Professor R. E. Langer, Director, Mathematics Research Center, U.S. Army, University of Wisconsin, 1118 W. Johnson Street, Madison 6, Wisconsin.

PERSONAL ITEMS

Associate Professor M. W. Milligan of Adams State College is on leave and has been awarded a Danforth Foundation Teacher Study Grant at Oklahoma State University.

Associated Colleges of Claremont, California: Harvey Mudd College, a new School of Science and Engineering, is a member of the Associated Colleges.

University of Kentucky: Dr. C. J. Scriba, University of Giessen, Germany, has been appointed Research Instructor; Professor H. H. Downing, formerly Head of the Department of Mathematics and Astronomy, has retired with the title of Professor Emeritus.

University of Pennsylvania: Mr. M. V. U. Krishna, University of Southern California, Mr. Kamini Patwary, Graduate Assistant at American University, and Mr. U. V. Ward, Graduate Student at Wayne State University, have been appointed Instructors; Assistant Professor George Butcher has been granted a National Science Foundation Fellowship and is on leave of absence during 1957–1958.

Assistant Professor Smbat Abian of the University of Tennessee has been appointed Assistant Professor at Queens College.

Associate Professor M. A. Al-Bassam, College of Engineering, Baghdad, Iraq, has been appointed Associate Professor at Lamar State College of Technology.

Dr. H. A. Antosiewicz of the National Bureau of Standards is on leave and has accepted a position as Editorial Consultant for Mathematical Reviews, American Mathematical Society, Providence, Rhode Island.

Mr. H. F. Bechtell, Jr., University of Wisconsin, has been appointed Assistant Professor at Grove City College.

Dr. Evelyn Boyd of the National Bureau of Standards has accepted a position as Research Mathematician for the Data Processing Center, Service Bureau Corporation, New York, New York.

Professor Leonard Bristow, University of Santa Clara, has been appointed Assistant Professor at San Jose State College.

Assistant Professor E. E. Capel, University of Miami, has accepted a position at Westinghouse Laboratories, Pittsburgh, Pennsylvania.

Associate Professor D. E. Carscallen, Wabash College, has retired.

Mr. W. H. Colbert, Jr., University of Nevada, has accepted a position as Engineer for Northrop Aircraft Inc., Hawthorne, California.

Mr. R. D. Depew has been transferred by IBM Corporation from Lincoln Laboratory, Lexington, Massachusetts, to Kingston, New York, as Associate Mathematician in Systems Development.

Dr. Gus DiAntonio, Bell Aircraft Corporation, has accepted a position as Senior Engineer for the Martin Company, Baltimore, Maryland.

Mr. H. L. Farris of Black, Sivalls & Bryson, Inc. is now Editorial Assistant for the Society of Exploration Geophysicists.

Mr. N. V. Fellers, Jr., American Institute for Foreign Trade, Phoenix, Arizona, is now associated with Ebasco International Corporation, New York, New York.

Mr. C. T. Fike, University of North Carolina, has accepted a position as Director of the ORACLE Applications Program, Oak Ridge Institute of Nuclear Studies, Oak Ridge, Tennessee.

Dr. L. R. Ford of RAND Corporation has accepted a position as Director of Operations Research, General Analysis Corporation, Sierra Vista, Arizona.

Mr. J. H. Griesmer, Princeton University, has accepted a position as Associate Mathematician for International Business Machines Research Center, Ossining, New York.

Dr. R. L. Helmbold, Carnegie Institute of Technology, has accepted a position as Operations Analyst with Technical Operations, Inc., Fort Monroe, Virginia.

Assistant Professor S. P. Hoffman, Jr., Polytechnic Institute of Brooklyn, has been appointed Assistant Professor at Trinity College, Connecticut.

Dr. M. L. Keedy, University of Nebraska, is now Associate Director of the Junior High School Mathematics Research Study, University of Maryland.

Professor B. C. Keeler, Webb Institute of Naval Architecture, has been appointed Dean of the Faculty.

Mr. C. D. Keim, previously a student at Gannon College, is now Technical Assistant at Lincoln Laboratory, Western Electric Corporation, Lexington, Massachusetts.

Dr. A. B. Lehman, Massachusetts Institute of Technology, has been appointed Research Assistant at Case Institute of Technology.

Dr. Nathaniel Macon of the General Electric Company has been appointed Associate Professor at Alabama Polytechnic Institute.

Associate Professor D. M. Merriell of Robert College has been appointed to an assistant professorship at Santa Barbara College, University of California.

Dr. D. M. Mesner, Purdue University, has accepted a position as Research Associate in the Statistical Engineering Laboratory, National Bureau of Standards, Washington, D. C.

Mr. Duncan Morrill, U. S. Army, has a position as Assistant Engineer for Western Electric Company, Winston-Salem, North Carolina.

Professor C. E. Moulton, Shurtleff College, has been appointed Associate Professor and Chairman of the Department of Mathematics, MacMurray College.

Mr. A. A. Mullin, Massachusetts Institute of Technology, has been appointed a member of the Mathematics Department of the Digital Computer Laboratory, University of Illinois.

Dr. G. M. Petersen, University College of Swansea, has been appointed to an associate professorship at the University of New Mexico.

Dr. R. P. Peterson, Jr., Operations Research, Bank of America, has accepted a position as Mathematician for the Matson Navigation Company, San Francisco, California.

Professor B. J. Pettis, Tulane University, has been appointed Professor at the University of North Carolina.

Professor Donald Everett Richmond, Williams College, is on leave and has been appointed Visiting Professor at Dartmouth College.

Mr. J. N. Rogers, University of Wisconsin, has accepted a position as Analyst with General Electric Company, Vallecitos Atomic Laboratory, Pleasanton, California.

Assistant Professor H. L. Rolf, Baylor University, has been appointed Assistant Professor at Georgetown College.

Dr. Berthold Schweizer, Illinois Institute of Technology, has been appointed Assistant Professor at San Diego State College.

Mr. Malcolm Smith, Motorola, Inc., has accepted a position as Senior Research Engineer at CONVAIR, Pomona, California.

Associate Professor M. R. Spiegel, Hartford Graduate Center, Rensselaer Polytechnic Institute, has been promoted to Professor.

Assistant Professor W. L. Strother, University of Miami, has been promoted to Associate Professor.

Associate Professor D. R. Sudborough, Central Michigan College, has been appointed Assistant Professor at San Jose State College.

Mr. John Todd, National Bureau of Standards, has been appointed Professor at the California Institute of Technology.

Dr. Bryant Tuckerman, Institute for Advanced Study, has accepted a position as Mathematician with the International Business Machines Corporation, Ossining, New York.

Dr. J. B. Tysver, University of Michigan, has accepted a position as Research Specialist at the Boeing Airplane Company, Seattle, Washington.

Dr. J. A. Ward has been made Chief of the Digital Computer Branch, Air Force Missile Development Center, Holloman Air Force Base, New Mexico.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 117 persons have been elected to membership by the Board of Governors on applications duly certified.

PAUL AIZLEY, A.B. (Harvard) Grad. Asst., University of Arizona.

SAMUEL I. ALTWERGER, B.S. (C.C.N.Y.) Head, Dept. of Math., Newton H.S., Elmhurst, N. Y.; New School of Social Research.

WILLARD E. ANDERSON, B.S. (Huron) Grad. Asst., South Dakota State College.

EDWARD J. BARBEAU, JR., Student, University of Toronto.

RALPH E. BEALS, Student, University of Kentucky.

JOHN A. BEEKMAN, M.S. (Iowa) Actuarial Dept., Minnesota Mutual Life Ins. Co., St. Paul, Minnesota.

WILLIAM F. BETHANY, B.S. (Memphis) Instr., Memphis State University.

WILLIAM C. BROWN, B.A. (Queens) Vice President & Actuary, The Colonial Life Ins. Co. of America, East Orange, New Jersey.

ORLANDO L. BYERS, JR., M.S. (Michigan) Chm., Dept. of Math., Edsel Ford H.S., Dearborn, Michigan.

MORTON CHALFIN, Computer, Sylvania Electric Products, Waltham, Massachusetts.

RUSSELL M. COLLINS, JR., B.S. (M.I.T.) Actuarial Dept., Minnesota Mutual Life Ins. Co., St. Paul, Minnesota.

HARVEY J. COOPERSMITH, B.S. (Brooklyn) Instr., Fairleigh Dickinson University.

CARMELO S. CRUZ, B.A. (Puerto Rico) Teacher, Juncos H.S., Juncos, Puerto Rico.

CHARLES H. CUNKLE, Ph.D. (Missouri) Asst. Professor, Colorado State University.

ANTHONY F. DAVIDOWSKI, M.Ed. (Pennsylvania S. U.) Instr., Mountain Lakes H.S., N. J.

CLAYTON W. DODGE, B.A. (Maine) Instr., University of Maine.

MRS. ELLEN H. DUNLAP, B.A. (California, Berkeley) Instr., Wayne State University.

ISTVAN FARY, Ph.D. (Sorbonne) Asso. Professor, University of Montreal.

LESTER J. FERGUSON, JR., M.A. (Temple) Instr., Fairleigh Dickinson University.

RUDOLF FITZ, B.S. (Oberrealschule Zwittau) Machinist, Denoyer & Geppert, Chicago, Illinois.

JAY E. FOLKERT, Ph.D. (Michigan S.U.) Head, Dept. of Math., Hope College.

EUGENE FRIEDMAN, B.S. (Southern Methodist) Quality Engineer, Temco Aircraft Corp., Dallas, Texas.

BERNARD GEDANKEN, M.A. (California) Asst. Professor, San Diego State College.

GLORIA GIOUMOUSIS, B.S. (Brooklyn) Technical Aide, Bell Telephone Laboratories, N. Y.

MARTIN GOLDSWORTH, M.S. (Alabama P.I.) Teacher, East Carolina College.

RONALD E. GRAF, Student, Rutgers University.

MALCOLM GRAHAM, Ed.D. (Columbia) Asst. Professor, University of Nevada.

JACK E. GRAVER, Student, Miami University.

RUPERT D. GRAVES, M.S. (Purdue) Instr., Brevard College.

NEWCOMB GREENLEAF, Student, Haverford College.

FRANCISCA GUZMAN, B.A. (Interamerican) Teacher, San German, Puerto Rico.

ALICE M. HACH, M.A. (Michigan) Consultant, Racine Public Schools, Wisconsin.

MYRON C. HAMER, M.A. (Columbia) Asst. Professor, Northeastern University.

JONATHAN G. HANKINS, JR., Student, University of Oklahoma.

ROBERT W. HARTOP, Student, University of Maine.

ERNEST HAWKINS, M.A. (Wesleyan) Professor, U. S. Naval Academy.

HAROLD A. HECKART, Ph.D. (Iowa) Head, Dept. of Physics and Math., Simpson College.

HARRY W. HICKEY, Pfc., U. S. Army, M.A. (Duke) Army Chemical Center, Maryland.

JAMES H. HOLLAND, D.Ed. (Calvin Coolidge) Head, Dept. of Math., Boston Technical H.S.

JEROME T. HOLLAND, A.B. (Whittier) Student, Albert-Ludwigs Universitat, Freiburg, Germany.

ROGER H. HOU, Student, Chung Chi College, Hong Kong.

DUANE M. JOHNSON, M.S. (North Dakota) Instr., Hamline University.

WOLODYMYR KALYNA, Ph.D. (Graz) Chm., Educational Council, Ukrainian Congress Committee, New York.

IRVING KAY, B.S. (C.C.N.Y.) Jr. Engineer, Burroughs Research Center, Paoli, Pa.

JERRY P. KING, Student, University of Kentucky.

WALTER E. KOSS, Ph.D. (Illinois) Assoc. Professor, Louisiana Polytechnic Institute.

HENRY J. KRATT, B.S. (Western Carolina) Grad. Student, University of Kentucky.

RICHARD G. LAATSCH, M.A. (Missouri) Instr., University of Tulsa.

PHILIP LAMB, M.A. (Oxford) Asst. Professor, Georgia State College.

JACK F. LATIMER, Student, University of California.

CHARLES R. LEGENDY, Student, Princeton University.

COL. (RET.) LAWRENCE C. LEONARD, B.S. (U.S.M.A.) Lecturer, College of William & Mary.

MARTIN LEVINE, M.A. (Pittsburgh) Asst. Professor, Pennsylvania State Univ.

MRS. ORDA E. LEWIS, B.A. (Los Angeles S.C.) Instr., Los Angeles State College.

MAURICE LIPTON, B.A. (Omaha) Systems Analyst-Math., Flight Simulation Lab., White Sands Proving Ground, New Mexico.

STUART P. LLOYD, Ph.D. (Illinois) Res. Math., Bell Telephone Labs., Murray Hill, N. J.

PAUL E. MAJOR, Student, Alabama Polytechnic Institute.

JAMES J. MARTYNIAK, B.S. (St. John Fisher) Grad. Asst., St. John's University, N. Y.

NICK L. MASSEY, B.A. (Hawaii) Head, Dept. of Math., Sealth High School, Seattle, Washington.

F. LOY MAUGH, Student, College of Idaho.

ELLSWORTH E. MAYER, M.S. (Kansas S.C.) Instr., Kansas State College.

JIMMY G. MERANDO, B.S. (Kansas S.T.C.) Teacher, Pittsburg, Kansas.

VICTOR J. MIZEL, Ph.D. (M.I.T.) Asst. Professor, University of Tennessee.

OTTO MOND, M.A. (Hunter) Math. Asst., U. S. Army Signal Engineering Lab., Woodhaven, N. Y.

JOHN C. MOORE, Ph.D. (Brown) Asso. Professor, Princeton University.

JOHN S. MOORE, JR., B.A. (Harding) Teacher, Florida Christian College.

THOMAS J. MORSE, B.A. (Yale) Student, Columbia University.

ARTHUR J. NADAS, Tool Engineer, Raytheon Mfg. Co., Bristol, Tennessee.

VINCENT H. NARAMORE, M.A. (Syracuse) Asso. Professor, St. Michael's Coll., Vermont.

CAPT. JOHN J. O'DONNELL, USN (Ret.), B.S. (U.S. Naval Academy) Silver Spring, Md.

REAR ADM. HERMAN O. PARISH, USN (Ret.), M.S. (Purdue) Lecturer, Virginia Military Institute.

GEORGE E. PARKER, B.S. (C.C.N.Y.) Instr., Fairleigh Dickinson University.

JOSEPH A. PAVELCAK, M.A. (Massachusetts) Instr., College of St. Thomas.

WALTER E. PEARSON, M.S. (Ball S.T.C.) Chairman, Technical Operations, St. Monterey, Calif.

MRS. EDYTHE M. PETIX, M.A. (Columbia) Chairman, Dept. of Math., Ridgefield Park H.S.

MARIO J. POLIFERNO, M.A. (Yale) Instr., Williams College.

JACK R. PORTER, Student, University of Oklahoma.

ELDEN E. PORTSCHY, B.S. (Nebraska) Teaching Asst., University of Nebraska.

JOHN RALEIGH, Ph.D. (Pennsylvania) Instr., University of Maryland.

MRS. PETRA G. RAMIREZ, B.S. (College of Agric.) Teacher, Dept. of Education, Mayaguez, Puerto Rico.

RAYMOND F. REES, A.B. (Brigham Young) Grad. Student, San Jose State Coll.; Engineer-in-charge, Electronic Defense Lab., Mountain View, Calif.

HENRY L. REEVES, M.A. (George Peabody) Asst. Professor, Memphis State Univ.

WARDELLE T. RICHARDSON, M.A. (Howard) Teacher, Frederick Douglass H.S., Baltimore, Md.

DONALD L. RICHTER, A.B. (Bowdoin) Res. Asst., Dept. of Statistics, University of North Carolina.

ERIC M. RICKARD, M.A. (Columbia) Teacher, Morristown School.

MARY C. ROGERS, B.S. (N.Y.U.) Chairman of Math., Roosevelt Jr. H.S., Westfield, N. J.

THOMAS DEF. ROGERS, JR., A.B. (Emory) Dept. of Math., Emory University.

FRANKLIN D. ROSCOE, B.S. (Idaho S.C.) Math., Westinghouse Electric Corp., Idaho Falls, Idaho.

CLAY C. ROSS, JR., Student, University of Kentucky.

M. J. SAADALDIN, B.A. (Higher Teachers T.C.) Grad. Student, Duke University.

BOBBY L. SANDERS, B.S. (East Texas S.C.) Grad. Student, Florida State University.

RICHARD L. SAUDER, Student, Virginia Military Institute.

WALTER W. SAWYER, M.A. (Cambridge) Asso. Professor, University of Illinois.

ROBERT P. SCHAEFER, Student, University of Houston; Accounting Clerk, Continental Oil Co., Houston, Texas.

ANNE E. SCHEERER, Ph.D. (Pennsylvania) Asst. Professor, Georgetown University.

FRANZ SCHNITZER, Ph.D. (Graz) Instr., Wayne State University.

JAMES W. SCHOMER, B.A. (San Jose S.C.) Engineering Asst., Stanford Research Inst.

JOHN W. SEWARD, B.Sc. (Reading) Applied Math., Harvard University.

HUGO E. SIEHR, M.E. (Marquette) Teacher, Bay City Junior College.

SISTER MARY GRACE, M.A. (Fordham) Teacher, Holy Family College, Pennsylvania.

JOHN H. SMITH, Ph.D. (Chicago) Chm., Dept. of Math. & Statistics, American Univ.

SAMMIE M. SMYRL, M.A. (Texas) Instr., Tyler Junior College.

THOMAS P. SODANO, Student, Indiana Technical College.

THEODORE G. STEWART, B.S. (North Carolina at Durham) Teacher, Reid H.S., Belmont, N.C.

VANCE D. STINE, M.A. (Southern Calif.) Instr., Los Angeles City College.

VIVIAN STRAND, M.A. (Iowa) Instr., Burlington College.

FRANCES L. STREET, M.S. (Tennessee) Asst. Professor, Memphis State University.

J. ALFRED STRINGHAM, B.S. (Syracuse) Mathematician, Griffiss A.F.B., Rome, N. Y.

BLAIR K. SWARTZ, B. S. (Antioch) Teaching Asst., Mass. Institute of Technology.

ANTHONY TRAMPUS, Ph.D. (Case) Sr. Engr., General Electric Co., Cincinnati, Ohio.

EDGAR P. VIRENE, Ph.D. (Pennsylvania) Asso. Professor, Newark Coll. of Engineering.

CAROL J. VON OHLEN, M.S. (Iowa) Instr., Iowa State T.C.

DOUGLAS S. WATTIER, B.A. (Carroll) Instr., Montana State College.

WALTER E. WIEBENSON, M.A. (California) Res. Math., Stanford Research Institute.

PAUL B. YALE, M.A. (Harvard) Teaching Fellow, Harvard University.

PETER YFF, Ph.D. (Illinois) Asst. Professor, Fresno State College.

NEAL ZIERLER, M.A. (Harvard) Member of Staff, Lincoln Laboratory, M.I.T.

THE DECEMBER MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The fall meeting of the Maryland-District of Columbia-Virginia Section was held at Georgetown University, Washington, D. C. on December 7, 1957 with 90 members in attendance. Professor R. P. Bailey, Chairman, presided at the meetings.

At the business session the Section considered the matter of inviting travelling lecturers to strengthen its programs. The Section voted to continue its Sectional Mathematics Contest in 1958.

The following papers were presented at the meeting:

1. *Report of the Commission on Mathematics*, by Executive Director A. E. Meder, Commission on Mathematics, New York, New York. (By invitation.)

See this MONTHLY, Report of the November (1957) Meeting of the New Jersey Section.

2. *On a "folk theorem" in probability theory*, by Professor Eugene Lukacs, The Catholic University of America, introduced by the Secretary.

E. J. McShane coined the term "folk theorem" to describe theorems which are known to many people through verbal communication but which have never been published with a formal proof. In this note we discuss such a theorem in probability theory. It is known that the characteristic function of every purely discrete distribution is almost periodic; the "folk theorem" to which we refer asserts that also the converse is true. We prove however a somewhat more restrictive statement: "A distribution function is purely discrete if, and only if, its characteristic function is almost periodic and has an absolutely convergent Fourier series." The validity of the folk theorem depends on the solution to the problem whether the characteristic function $f(t)$ of a singular distribution can be an almost periodic function.

3. *The Maryland-District of Columbia-Virginia Mathematics Contest*, by Professor D. B. Lloyd, District of Columbia Teachers College.

On account of an elaborate award system which it has built up, this section will repeat its own contest program this year. This is its fourth year of successful operation. However the Section will use the test questions of the National Contest Committee and administer the test on the same date. Awards consisting of Savings Bonds, Clock Radios, Slide Rules, and other small items, are contributed by local concerns. Both local, state, and over-all awards are made, as well as certificates to the winner at each school. Committee members are: W. H. Norris, Chairman; Ferrell Atkins, P. L. Chessin, Bro. Dunston Hayden, S. B. Jackson, Fr. Koehler, D. B. Lloyd, Carol McCamman, K. Meals, W. G. Youden.

4. *Vertices of space curves*, by Professor C. S. Wolfe, U. S. Naval Academy.

The purpose of this paper is to show the relationship between various definitions of a vertex and give a necessary and sufficient condition for the existence of a vertex. On a space curve C satisfying certain conditions a point P is defined to be a vertex if and only if the osculating sphere at P has contact of order at least four with C at P. An interesting fact is that this definition is not equivalent to the condition that the radius of the osculating sphere be an extreme value at P. An example is given and the corresponding results in two dimensions are derived.

5. *An equivalent linearization solution of a fourth-order nonlinear system*, by Dr. C. H. Murphy, Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland.

The usual approach to the solution of a fourth-order nonlinear system is handicapped by tedious algebraic manipulations and the fact that the associated phase plane is four dimensional. For the nontrivial case of members of the epicyclic subset, the equivalent linearization technique can be simply applied through the use of a complex variable. The existence of limit motions and the dependence of the type of motion on initial conditions can then be easily studied by means of a two-dimensional "amplitude plane."

6. *Nomographic solutions for position relations between a close earth satellite and its observer*, by Dr. R. H. Wilson, Jr., Naval Research Laboratory, Washington, D. C.

The ground range and azimuth of a close earth satellite are related to the observer's and satellite's latitudes and difference of longitude by the spherical trigonometrical solutions of the astronomi-

cal triangle, for which tables and nomograms are available. Additional alignment charts have been devised for the apparent angular elevation as related to the ground range and linear height of the satellite above the earth; also for its slant range as related to ground range and elevation. Such charts are especially desirable for routine use by nontechnical observers of artificial satellites.

D. B. LLOYD, *Secretary*

CONTINUATION OF THE PROGRAM OF VISITING LECTURERS

The recent grant of the National Science Foundation permits the continuation of this program through the academic year 1958–59.

The Association's Committee on Visiting Lecturers consists of Professors P. B. Johnson, B. W. Jones, D. E. Richmond, Rothwell Stephens and R. A. Rosenbaum, Chairman. The Committee is authorized to select lecturers and to arrange their itinerary. Any correspondence on either of these topics should be addressed to Professor Rosenbaum as Chairman of the Committee.

CALENDAR OF FUTURE MEETINGS

Thirty-ninth Summer Meeting, Massachusetts Institute of Technology, Cambridge, Massachusetts, August 25–28, 1958.

Forty-second Annual Meeting, University of Pennsylvania, Philadelphia, Pennsylvania, January 22–23, 1959.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Washington and Jefferson College, Washington, Pennsylvania, May 3, 1958.

ILLINOIS, Illinois College, Jacksonville, May 9–10, 1958.

INDIANA, Ball State Teachers College, Muncie, May 3, 1958.

IOWA, Drake University, Des Moines, April 18, 1958.

KANSAS, Kansas State Teachers College, Emporia, April 12, 1958.

KENTUCKY, University of Kentucky, Lexington, April 26, 1958.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Randolph-Macon Woman's College, Lynchburg, Virginia, April 26, 1958.

METROPOLITAN NEW YORK, Hofstra College, Hempstead, New York, April 19, 1958.

MICHIGAN, University of Michigan, Ann Arbor, March 22, 1958.

MINNESOTA, St. John's University, Collegeville, Minnesota, May 17, 1958.

MISSOURI, University of Missouri, Columbia, May 3, 1958.

NEBRASKA, University of Nebraska, Lincoln, April 19, 1958.

NEW JERSEY, Rutgers University, New Brunswick, November 1, 1958.

NORTHEASTERN

NORTHERN CALIFORNIA, University of California, Berkeley, June 17, 1958 (joint meeting with ASEE, mathematics division).

OHIO, Denison University, Granville, April 26, 1958.

OKLAHOMA, Central State College, Edmond, April 18–19, 1958.

PACIFIC NORTHWEST, Oregon State College, Corvallis, June 20, 1958.

PHILADELPHIA

ROCKY MOUNTAIN, Colorado State College, Greeley, May 9–10, 1958.

SOUTHEASTERN, University of Florida, Gainesville, March 14–15, 1958.

SOUTHERN CALIFORNIA, Pasadena City College, March 8, 1958.

SOUTHWESTERN, University of New Mexico, Albuquerque, April 11–12, 1958.

TEXAS, Baylor University, Waco, April 18–19, 1958.

UPPER NEW YORK STATE, École Polytechnique and University of Montreal, Montreal, Quebec, Canada, May 10, 1958.

WISCONSIN, Carroll College, Waukesha, May 3, 1958.

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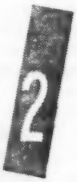

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